5. NON-STRICTLY CONVEX DOMAINS

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neighbourhood U_ε of 0 such that $f(x) < \varepsilon|x|$ in this neighbourhood. But in view of $f_{x_i}(0) = 0$, $i = 1, \ldots, n$, we have $f(0, \ldots, 0, x_i, 0, \ldots, 0) < \varepsilon|x_i|$ for sufficiently small x_i and hence by convexity $f(x) < \varepsilon|x|$ for sufficiently small $|x|$. □

REMARK 4.3. The following was announced in [B00]: *If a strictly convex domain D is divisible, that is, if it admits a proper cocompact group of isometries Γ, then D is Gromov hyperbolic if and only if ∂D is C^1.* Our Theorem 4.2 shows that in the implication (Gromov hyperbolicity + divisibility $\Rightarrow C^1$) the condition of divisibility is superfluous.

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This section owes much of its existence to [Be97] and [Be99]. Using a different argument, we prove certain extensions to arbitrary convex bounded domains of some of the results obtained in those papers.

LEMMA 5.1. *Let D be a bounded convex domain in $\mathbf{R}^n$. Let $\{x_n\}, \{y_n\}$ be two sequences of points in D. Assume that $x_n \to \bar{x} \in \partial D$, $y_n \to \bar{y} \in \bar{D}$ and $[\bar{x}, \bar{y}] \not\subseteq \partial D$. Let x_n' and y_n' denote the endpoints of the chord through x_n and y_n as usual. Then x_n' converges to $\bar{x}$ and y_n' converges to the endpoint $\bar{y}'$ of the chord defined by $\bar{x}$ and $\bar{y}$ different from $\bar{x}$.*

Proof. Compare with Lemma 5.3. in [Be97]. Every limit point of chord endpoints must belong to the line through $\bar{x}$ and $\bar{y}$. In addition, in the case of x_n' for example, any limit point must lie on the halfline from $\bar{x}$ not containing $\bar{y}$. At the same time each limit point must belong to the boundary of D, and the statement follows since the line through $\bar{x}$ and $\bar{y}$ intersects ∂D only in $\bar{x}$ and $\bar{y}'$. □

THEOREM 5.2. *Let D be a bounded convex domain. Let $\{x_n\}$ and $\{z_n\}$ be two sequences of points in D. Assume that $x_n \to \bar{x} \in \partial D$, $z_n \to \bar{z} \in \partial D$ and $[\bar{x}, \bar{z}] \not\subseteq \partial D$. Then there is a constant $K = K(\bar{x}, \bar{z})$ such that for the Gromov product $(x_n \mid z_n)_y$ in Hilbert distances relative to some fixed point y in D we have*

$$\limsup_{n \to \infty} (x_n \mid z_n)_y \leq K .$$

Proof. By Lemma 5.1, the endpoints of the chords through x_n and z_n converge to $\bar{x}$ and $\bar{z}$.

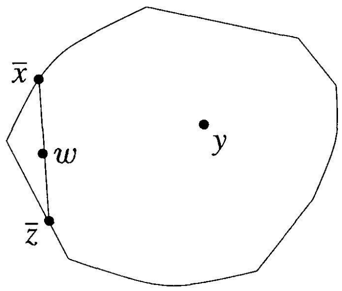

FIGURE 5
Partial hyperbolicity

Since $[\bar{x}, \bar{z}]$ is not contained in the boundary of D, there are small compact neighbourhoods $\overline{U}_x$ and $\overline{U}_z$ of $\bar{x}$ and $\bar{z}$ respectively, in ∂D, such that every chord with endpoints in $\overline{U}_x$ and $\overline{U}_z$ is contained in D. In particular the Euclidean midpoint of every such chord lies inside D and by compactness there is an upper bound K on $h(y, w)$, where w is the midpoint of such a chord.

Consider three points x, y, z and a point w on a (minimizing) geodesic segment $[x, z]$ in a (geodesic) metric space (Y, d). Then

$$\begin{aligned}(x \mid z)_y &= \tfrac{1}{2}(d(x,y) + d(z,y) - d(x,z)) \\ &= \tfrac{1}{2}(d(x,y) + d(z,y) - d(x,w) - d(w,z)) \\ &\leq \tfrac{1}{2}(d(y,w) + d(y,w)) = d(y,w)\end{aligned}$$

by the triangle inequality. It follows from this estimate and the above considerations that eventually

$$(x_n \mid z_n)_y \leq K. \qquad \square$$

REMARK 5.3. The content of Theorem 5.2 is that (D, h) satisfies a weak notion of hyperbolicity. This property should be compared with Gromov hyperbolicity, especially with the fact that for Gromov hyperbolic spaces, two sequences converge to the same point of the boundary if and only if their Gromov product tends to infinity. Theorem 5.2 can be applied as in [Ka01, Theorem 8] to the study of random walks on the automorphism group of D, and it is also likely to be useful for analyzing commuting nonexpanding maps or isometries of (D, h).

REMARK 5.4. We suggest that a similar statement might hold for the classical Teichmüller spaces and perhaps also for more general Kobayashi hyperbolic complex spaces. Hilbert geodesic rays from a point y that terminate on a line segment contained in the boundary may correspond to the Teichmüller geodesic rays defined by Jenkins-Strebel differentials that H. Masur considered when demonstrating the failure of CAT(0) for the Teichmüller space of Riemann surfaces of genus $g \geq 2$. The complement of the union of all line segments in the boundary ∂D may correspond to the uniquely ergodic foliation points on the Thurston boundary of Teichmüller space.

Using the arguments in [Ka01], see Proposition 5.1 of that paper, we obtain the following result as an application of Theorem 5.2:

THEOREM 5.5. *Let D be a bounded convex domain and $\varphi\colon D \to D$ be a map which does not increase Hilbert distances. Then either the orbit $\{\varphi^n(y)\}_{n=1}^{\infty}$ is bounded or there is a limit point $\bar{y}$ of the orbit such that for any other limit point $\bar{x}$ of the orbit it holds that $[\bar{x}, \bar{y}] \subset \partial D$.*

This theorem, which extends a theorem in [Be97], provides a general geometric explanation for a part of the main theorem in [Me01].

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