Introduction

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SEMISTABLE $K3$-SURFACES WITH ICOSAHEDRAL SYMMETRY

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ABSTRACT. In a Type III degeneration of $K3$-surfaces the dual graph of the central fibre is a triangulation of S^2. We realise the tetrahedral, octahedral, and especially the icosahedral triangulation in families of $K3$-surfaces, preferably with the associated symmetry groups acting.

INTRODUCTION

A degeneration of surfaces is a 1-parameter family with general fibre a smooth complex surface. The case of $K3$-surfaces has attracted a great deal of attention. A nice discussion is contained in the introductory first paper [F-M] of the bundle [SAGS]. One usually allows base change and modifications to obtain good models. After a ramified cover of the base and resolution of singularities we may assume that the degeneration $f\colon \mathcal{X} \to S \ni 0$ is *semistable*: the zero fibre $X = f^{-1}(0)$ is a reduced divisor with (simple) normal crossings in the smooth manifold $\mathcal{X}$. Further modifications of a $K3$-degeneration lead to a minimal model, which falls into one of three types.

In a Type III degeneration of $K3$-surfaces the dual graph of the central fibre is a triangulation of S^2. In this paper I construct an example with my favourite triangulation, the icosahedral one. A substantial part is taken up by the tetrahedral case, which is easier to handle and allows more explicit results. A second purpose of this paper is to link general theory with concrete computations.

There are two obvious ways to realise a semistable degeneration with prescribed combinatorial type. The first is to start with a singular total space,

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but correct central fibre. An example is the coordinate tetrahedron $T = x_0x_1x_2x_3$ in $\mathbf{P}^3$, where we get a degeneration $f\colon \mathcal{X} \to \mathbf{P}^1 \ni 0$ by blowing up the base locus of the pencil spanned by T and a generic quartic. The total space then has 24 A_1-singularities. This is a minimal model in the Mori category, but one has to take a small resolution to get a smooth total space. We can arrange that each plane of the tetrahedron is blown up in 6 points. The dual graph of the central fibre remains the same.

The second method is to try to smooth the putative central fibre. For the tetrahedron this can be done directly. We glue together four cubic surfaces along triangles. This normal crossings variety satisfies the topological conditions to be a central fibre (the triple point formula), but one also needs a more subtle analytic condition (d-semistability), which translates into equations on the coefficients in the equations. The necessary deformation theory in general has been developed by Friedman [F2]. The central result is that smoothing is always possible in the $K3$-case. This holds both for abstract and embedded deformations. One typically obtains different degenerations from the two constructions, which fill up 19 dimensional families in a 20 dimensional deformation space of the normal crossings $K3$.

For the octahedron both methods can again be applied. We find the correct space as anticanonical divisor in the toric threefold given by a cube. In his monograph Ulf Persson invites the reader 'to find a degeneration into a dodekahedron of rational surfaces' [P, p. 126]. For a construction according to the first method the double curve must be an anti-canonical divisor on each component. The most natural choice for a rational surface is then a Del Pezzo of degree five. This makes that the dodecahedron itself has degree 60, and it is exactly such a dodecahedron, obtained by gluing twelve Del Pezzo surfaces, which the second method smoothes. Unfortunately the computations are too difficult to give explicit formulas. The same holds for a related problem in fewer variables, smoothing the Stanley-Reisner ring of the icosahedron. A semistable model of such a degeneration has as central fibre a complexified football. The extra components come from singularities of the total space.

The last example suggests that one can get a dodecahedron out of a central fibre with fewer than 12 components. This requires a breaking of the symmetry. By combining the first and the second method I obtain in 5.12 an explicit degeneration, whose general fibre is a smooth $K3$-surface of degree 12 in $\mathbf{P}^7$, with special fibre consisting of 6 planes with triangles as double curves and 3 quadric surfaces with rectangular double curve. Its total space has three singularities, which are isomorphic to cones over Del Pezzo surfaces of degree 5, and 18 A_1 points. The dual graph of the central fibre on a suitable smooth model is the icosahedron.

This paper is organised as follows. In the first section I recall the results on degenerations of $K3$-surfaces, in particular that one can always realise a particularly nice model, the (-1)-form. Section 2 brings as illustration detailed computations for tetrahedra. The results fit in with the general deformation theory, which is treated in the third section, with special emphasis on degenerations in (-1)-form. A short fourth section introduces the combinatorial tools to handle large systems of equations: the definitions of Stanley-Reisner rings and Hodge algebras are reviewed. The final section contains the dodecahedral degenerations.

1. Semistable degenerations of $K3$-surfaces

1.1. The name $K3$ has been explained by André Weil: « en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne $K2$ au Cachemire » [W, p. 546]. He calls any surface a $K3$, if it has the differentiable structure of a smooth quartic surface in $\mathbf{P}^3(\mathbf{C})$. A Kummer surface is a quartic with 16 A_1-singularities. As these singularities admit simultaneous resolution, the minimal resolution of a Kummer surface deforms into a smooth quartic and is therefore a $K3$-surface. A quartic surface X is simply connected, so in particular $b_1(X) = 0$ and has trivial canonical sheaf by the adjunction formula: X is an anti-canonical divisor in $\mathbf{P}^3$. The modern definition of a $K3$-surface: $b_1(X) = 0$ and $K_X = 0$, is equivalent with Weil's definition because all $K3$-surfaces form one connected family.

1.2. Let $f\colon \mathcal{X} \to S \ni 0$ be a proper surjective holomorphic map of a 3-dimensional complex manifold $\mathcal{X}$ to a (germ of a) curve S such that the zero fibre $X = f^{-1}(0)$ is a reduced divisor with (simple) normal crossings; then the degeneration f is called *semistable.*

In the $K3$ case the following holds (see [F-M] for exact references):

1.3. Theorem (Kulikov). *Let $f\colon \mathcal{X} \to S$ be a semistable degeneration of $K3$-surfaces. If all components of $X = f^{-1}(0)$ are Kähler, then there exists a modification $\mathcal{X}'$ of $\mathcal{X}$ such that $K_{\mathcal{X}'} \equiv 0$.*

A degeneration as in the conclusion of the theorem ($K_{\mathcal{X}} \equiv 0$) is called a *Kulikov model.*