Zeitschrift: L'Enseignement Mathématique

Band: 49 (2003)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TILE HOMOTOPY GROUPS

Kapitel: 6. Criteria for $\pi(\Tau)$ to be abelian

Autor: Reid, Michael

DOI: https://doi.org/10.5169/seals-66684

6. Criteria for $\pi(\mathcal{T})$ to be abelian

In many cases that we have examined, the tile homotopy group turns out to be abelian. In such cases, the tile homotopy group gives no further information than the tile homology group, which is generally more accessible. We give here two general criteria which imply that $\pi(\mathcal{T})$ is abelian.

THEOREM 6.1. *Suppose that the set of prototiles $\mathcal{T}$ is rotationally invariant.*

(a) *If $\bar{x}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ in $P(\mathcal{T})$, then $\pi(\mathcal{T})$ is cyclic, and its order is the greatest common divisor of the sizes of tiles in $\mathcal{T}$. If d is this greatest common divisor, then a specific isomorphism $\pi(\mathcal{T}) \xrightarrow{\cong} \mathbf{Z}/d\mathbf{Z}$ is given by $[\gamma] \mapsto N \bmod d$, where the loop γ encloses N squares, counting multiplicity.*

(b) *If $\bar{x}\bar{y}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ in $P(\mathcal{T})$, then $\pi(\mathcal{T})$ is abelian. Let $H \subseteq \mathbf{Z}^2$ be the subgroup generated by all elements of the form (b,r) and (r,b), where there is a tile in $\mathcal{T}$ with b black squares and r red squares. Then $\pi(\mathcal{T}) \cong \mathbf{Z}^2/H$, and a specific isomorphism is given by $[\gamma] \mapsto (B,R) \bmod H$, where the loop γ encloses B black squares and R red squares, counting multiplicity.*

Proof. (a) A $90°$ clockwise rotation corresponds to mapping x and y to y^{-1} and x respectively. Since $\mathcal{T}$ is invariant under this rotation, this map induces an automorphism of $P(\mathcal{T})$. Thus $\bar{y}^{-1}$ commutes with $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}^{-1}$, and therefore also with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Now $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ is central in $P(\mathcal{T})$. We have seen that $\pi(\mathcal{T})$ is generated by the elements $\bar{c}_{ij} = \bar{x}^i\bar{y}^j\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}\bar{y}^{-j}\bar{x}^{-i}$, and our commutativity relations show that these are all equal to $\bar{c} = \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Thus $\pi(\mathcal{T})$ is generated by a single element, $\bar{c}$, and therefore is cyclic.

Let $w \in C$ be the boundary word of a tile in $\mathcal{T}$, which imposes a relation upon $P(\mathcal{T})$. Then w can be written uniquely as a word in the elements c_{ij}. The total weight in an individual c_{ij} is the winding number around square (i,j), which is either 1 or 0, according to whether or not that square is in the tile. Thus the total weight in all the c_{ij}'s is the size of the tile. Therefore, w imposes the relation $\bar{c}^n = 1$ on $\pi(\mathcal{T})$, where n is the size of the tile. The remainder of the statement is now clear.

(b) Since $\bar{x}\bar{y}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, so does $\bar{x}^{-1}\bar{y}^{-1}$. A $90°$ clockwise rotation shows that $\bar{y}^{-1}\bar{x}$ commutes with $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}^{-1}$, and conjugating by $\bar{y}$ shows that $\bar{x}\bar{y}^{-1}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Now we see that both $\bar{x}^2 = (\bar{x}\bar{y}^{-1})(\bar{x}^{-1}\bar{y}^{-1})^{-1}$ and $\bar{y}^2 = (\bar{x}^{-1}\bar{y}^{-1})^{-1}(\bar{x}\bar{y}^{-1})^{-1}$ also commute with

$\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Next, $\pi(\mathcal{T})$ is generated by the elements $\bar{c}_{ij}$. Our commutativity relations show that $\bar{c}_{ij} = \bar{c}_{00}$ if $i+j$ is even, while $\bar{c}_{ij} = \bar{c}_{10}$ if $i+j$ is odd. Moreover, these two elements commute with each other, because $\bar{c}_{00} = \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, and $\bar{c}_{10} = (\bar{x}^2)(\bar{x}\bar{y}^{-1})^{-1}(\bar{x}\bar{y})^{-1}$.

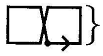
Let $w \in C$ be the boundary word of a tile in $\mathcal{T}$, which may be written uniquely as a word in the elements c_{ij}. The total weight in those c_{ij}'s with $i+j$ even [respectively, odd] is the number of black [respectively, red] squares in this placement of the tile. Thus w imposes the relation $\bar{c}_{00}^b\bar{c}_{10}^r = 1$ on $\pi(\mathcal{T})$, and the relation $\bar{c}_{00}^r\bar{c}_{10}^b = 1$ comes from the boundary word xwx^{-1}. The statement now follows. □

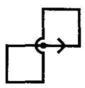
It may be useful to reformulate Theorem 6.1 in a different way. We will consider the following self-intersecting closed paths to depict "generalized tiles" that have boundary words $xyx^{-1}y^{-1}x^{-1}yxy^{-1}$ and $xyx^{-1}y^{-2}x^{-1}yx$ respectively.

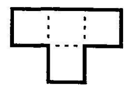
FIGURE 6.2
Generalized tiles

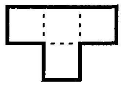
Now Theorem 6.1 may be rephrased as follows.

THEOREM 6.3. *Suppose that rotations are allowed in our protosets.*

(a) *The tile homotopy group of* $\mathcal{T} = \{$$\}$ *is isomorphic to* $\mathbf{Z}$, *and a specific isomorphism is given by* $[\gamma] \mapsto N$, *where the loop* γ *encloses* N *squares, counting multiplicity.*

(b) *The tile homotopy group of* $\mathcal{T} = \{$$\}$ *is isomorphic to* $\mathbf{Z}^2$, *and a specific isomorphism is given by* $[\gamma] \mapsto (B,R)$, *where the loop* γ *encloses* B *black squares and* R *red squares, counting multiplicity.* □

Conway and Lagarias mention the protoset $\mathcal{T} = \{$$\}$, with all orientations allowed. They remark that Walkup [17] has shown that if an $m \times n$ rectangle can be tiled by $\mathcal{T}$, then both m and n are multiples of 4. They also note that a rectangle has a signed tiling by $\mathcal{T}$ if and only if its area is a multiple of 8. They implicitly ask what the relationship between Walkup's proof and the tile homotopy method is. Theorem 6.1 above allows us to compute the tile homotopy group of $\mathcal{T}$.

PROPOSITION 6.4. *The tile homotopy group of* $\mathcal{T} = \{$$\}$ *is* $\mathbf{Z}/8\mathbf{Z}$. *A specific isomorphism is given by* $[\gamma] \mapsto (B + 5R) \bmod 8$, *where the loop* γ *encloses* B *black squares and* R *red squares, counting multiplicity.*

Proof. The boundary words of the orientations

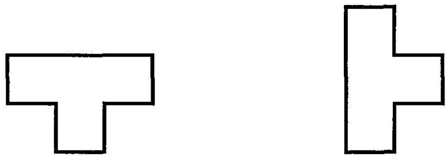
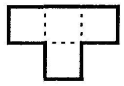


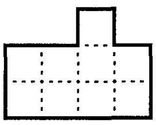
FIGURE 6.5
Two orientations of the T tetromino

give the relations $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}^{-3}\bar{y}^{-1}\bar{x} = 1 = \bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}^{-1}\bar{y}\bar{x}^{-1}\bar{y}^{-2}$ in $P(\mathcal{T})$. Therefore, $\bar{x}^{-2}\bar{y}^{-1}\bar{x} = \bar{y}\bar{x}^{-1}\bar{y}^{-2}$, which is equivalent to $\bar{x}\bar{y}$ commuting with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Now part (b) of Theorem 6.1 shows that $\pi(\mathcal{T}) \cong \mathbf{Z}^2/\langle(1,3),(3,1)\rangle \cong \mathbf{Z}/8\mathbf{Z}$, and the specific isomorphism is as claimed. $\square$

COROLLARY 6.6. *The boundary word of a rectangle is trivial in* $\pi(\{$$\})$ *if and only if its area is divisible by* 8. $\square$

This shows that Walkup's proof is unrelated to tile homotopy; his proof relies on subtle geometric restrictions that are not detected by the tile homotopy group.

Another example that exhibits a similar phenomenon in a more obvious manner is the following.

EXAMPLE 6.7. Let $\mathcal{T} = \{$$\}$, with all orientations allowed. Comparing the two orientations

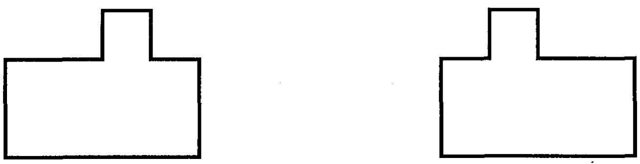


FIGURE 6.8
Two orientations of a tile

shows that $\bar{x}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ in the tile path group. Then Theorem 6.1 (a) shows that $\pi(\mathcal{T}) \cong \mathbf{Z}/9\mathbf{Z}$. This means that the tile homotopy group only detects area, modulo 9.

On the other hand, we can easily show that if $\mathcal{T}$ tiles a rectangle, then both sides must be even. Consider the ways that a tile can touch the edge of a rectangle.

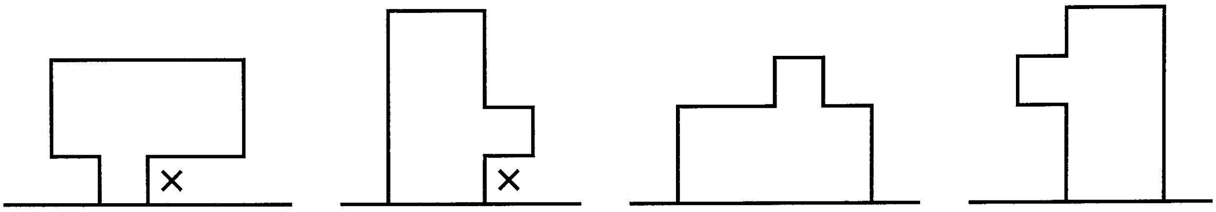
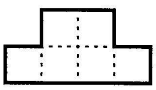


FIGURE 6.9
Tiles along an edge of a rectangle

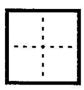
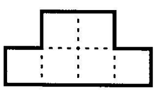
We see that the first two possibilities cannot occur, so each tile that touches the edge does so along an even length. Therefore, each edge of the rectangle has even length. In fact, it is not much harder to show that if $\mathcal{T}$ tiles an $m \times n$ rectangle, then both m and n are multiples of 6. A straightforward argument shows that every tiling of a quadrant by $\mathcal{T}$ is a union of 6×6 squares, which implies the result.

7. APPENDIX: FURTHER EXAMPLES

Here we give some more tiling restrictions we have found using the tile homotopy technique. In each case, there are signed tilings that show that the result cannot be obtained by tile homology methods, and there are tilings that show that the result is non-vacuous. Further details will be published elsewhere.

THEOREM 7.1. *Let* $\mathcal{T} = \{$  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then either* m *or* n *is a multiple of* 4.
(b) *A* 1×6 *rectangle has a signed tiling by* $\mathcal{T}$.

THEOREM 7.2. *Let* $\mathcal{T} = \{$ ,  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then* mn *is a multiple of* 4.
(b) *A* 1×6 *rectangle has a signed tiling by* $\mathcal{T}$.