SYMPLECTIC LOOK AT SURFACES OF REVOLUTION

L'Enseignement Mathématique, t. 49 (2003), p. 157–172

A SYMPLECTIC LOOK AT SURFACES OF REVOLUTION

by Andrew D. HWANG

Dedicated to Professor Shoshichi Kobayashi on his 70th birthday

1. INTRODUCTION

Informally, a surface of revolution is a 2-dimensional Riemannian manifold Σ equipped with an isometric circle action. Surfaces of revolution are among the simplest objects in differential geometry; the metric is determined by a single function of one real variable, hence can be specified by solving an *ordinary* differential equation.

A function $x: \Sigma \to \mathbf{R}$ is an "orbit parameter" if each level set of x is a single orbit. Given an orbit parameter, a "profile" for Σ is a function that determines the lengths of the orbits. For example, when the graph of a function ξ is revolved about an axis in $\mathbf{R}^3$, the "obvious" orbit parameter is a Cartesian coordinate x along the axis of revolution, and ξ itself is a profile.

This note constructs surfaces of revolution from an elementary, intrinsic orbit parameter and profile function. The point of departure is a theorem of Archimedes, whose proof is nowadays an easy calculus exercise. Let $S \subset \mathbf{R}^3$ be the unit sphere, regarded as a surface of revolution by fixing an arbitrary diameter. A "zone" of S is a subset bounded by two planes perpendicular to the diameter, and the "height" of a zone is the distance between its bounding planes.

THEOREM 1.1. *A zone of height h on the unit sphere has area $2\pi h$; in particular, the area depends only on the height of the zone.*

To reformulate, let x be a Cartesian coordinate along the diameter, and let O be the equator $\{x = 0\}$. Each $p \in S$ lies on a unique circle O_p perpendicular to the diameter; let $2\pi\tau$ be the oriented area bounded by O and O_p. Theorem 1.1 asserts that the extrinsic position x and the intrinsic coordinate τ are *the same orbit parameter*.

Of course, distance along the axis of rotation does not correspond so nicely with zonal area on a general surface of revolution, but in some ways area is a "better" parameter: with a judicious choice of profile function, the Gaussian curvature becomes extremely simple. The resulting description makes it easy to study and classify surfaces of revolution that have specified Gaussian curvature. The motivation for this description comes from symplectic and Kähler geometry, but the idea and methods are elementary.

2. Abstract surfaces of revolution

Identify the circle S^1 with the multiplicative group of complex numbers of norm 1, and let $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere, equipped with the S^1-action induced by multiplication on $\mathbf{C}$. In this note, an *abstract surface of revolution* is a pair $\Sigma = (D, g)$ consisting of a connected, S^1-invariant domain $D \subset \mathbf{P}^1$ and an S^1-invariant metric g, possibly with conical singularities at the fixed points.

General metrics in coordinates

There are two "natural" coordinate systems on an abstract surface of revolution: *isothermal parameters* adapted to the circle action, and *action-angle coordinates*. While each highlights aspects of the metric geometry, their interplay is synergistic, and naturally suggests the "correct" choice of profile.

Isothermal parameters. A coordinate system (x, y) is said to be *isothermal* for the metric g if there exists a (locally defined) function $\psi = \psi(x, y)$ such that

$$g = e^{\psi}(dx^2 + dy^2).$$

On a surface of revolution, existence of isothermal parameters is elementary. To wit, choose local coordinates (r, θ) in which $\frac{\partial}{\partial\theta}$ generates the S^1 action. Because the metric is invariant under the circle action, the components of g

do not depend on θ. Fix a point $w_0 \in D$ and consider the curve through w_0 that is everywhere g-orthogonal to the S^1 orbits. Let s be a real coordinate along this curve, and use the circle action to extend s to all of D; in (s, θ), the metric has the form

$$g = e^{\psi_1(s)}\, ds^2 + e^{\psi_2(s)}\, d\theta^2 .$$

Now solve the differential equation $e^{\psi_1(s)} s'(t)^2 = e^{\psi_2(s)}$ for s as a function of t, and set $\psi = \psi_2 \circ s$. In (t, θ), the metric has the form

$$g = e^{\psi(t)}\left(dt^2 + d\theta^2\right) . \tag{2.1}$$

THE AREA FORM AND ACTION-ANGLE COORDINATES. By (2.1), the area element of g is the 2-form

$$dA = e^{\psi(t)}\, dt \wedge d\theta .$$

Writing $d\tau$ for the exact 1-form $e^{\psi(t)}\, dt$, the area form is

$$dA = d\tau \wedge d\theta . \tag{2.2}$$

The function τ, unique up to an additive constant, is a function of t alone, i.e., is constant on the orbits of the S^1 action.

A *zone* of Σ is a connected region bounded by two orbits. Equation (2.2) immediately implies that the zone $\{\tau_0 \leq \tau \leq \tau_1\}$ has area $2\pi(\tau_1 - \tau_0)$ for all $\tau_0 < \tau_1$. In symplectic geometry, an S^1-invariant function with this property is called a *moment map* of the circle action, and (τ, θ) are called *action-angle* variables. Archimedes' theorem asserts that for the unit sphere in $\mathbf{R}^3$, projection to a diameter is a moment map for the circle action that revolves the sphere about that diameter.

Introducing the function $\varphi(\tau) = e^{\psi(t)}$, the metric g is given by

$$\varphi(\tau)\left(dt^2 + d\theta^2\right) = \frac{1}{\varphi(\tau)}\, d\tau^2 + \varphi(\tau)\, d\theta^2 . \tag{2.3}$$

The thesis of this note is that φ, henceforth called the *momentum profile* of the metric, is the correct choice of profile for investigations concerning Gaussian curvature.

Equation (2.3) implies that the length element along an orbit is $\sqrt{\varphi(\tau)}\, d\theta$, so the length of an orbit is $2\pi\sqrt{\varphi(\tau)}$. Similarly, the arc length element along a generator of the surface is

$$ds = \sqrt{\varphi(\tau)}\, dt = \frac{d\tau}{\sqrt{\varphi(\tau)}} . \tag{2.4}$$

The geometry (in almost the literal sense of "earth measurement") of the metric is depicted in Figure 2.1. Fix an orbit O and define a function $\tau: D \to \mathbf{R}$ by letting $2\pi\tau(p)$ be the oriented area of the zone bounded by O and the orbit through p. Let $I \subset \mathbf{R}$ be the image of τ, and define the non-negative function $\varphi: I \to \mathbf{R}$ so that $2\pi\sqrt{\varphi(\tau)}$ is the length of the orbit through p.

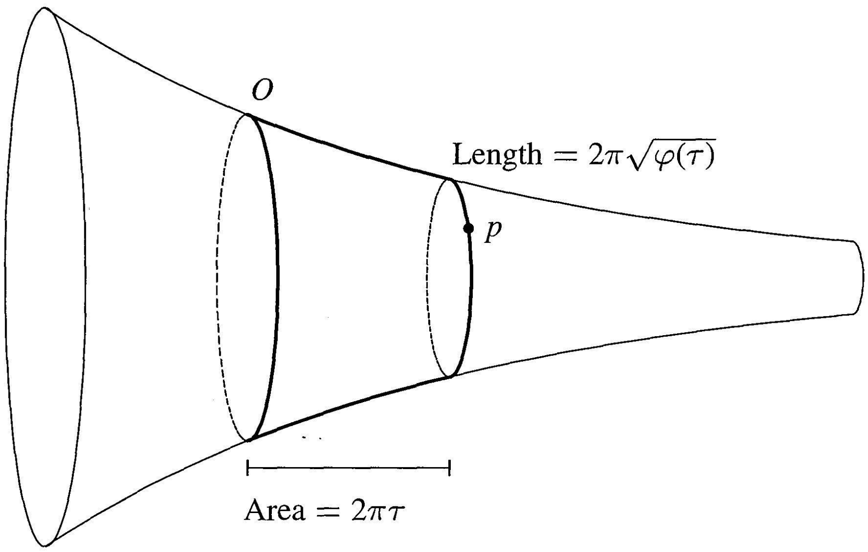

FIGURE 2.1
A metric in terms of zonal area

CONSTRUCTING THE METRIC

Figure 2.1 expresses the moment map τ and the momentum profile φ in terms of the metric geometry. In order to work analytically with surfaces of revolution, it is desirable to reverse this development. Clearly, g can be recovered from τ and φ; remarkably, φ alone is enough.

In $\mathbf{P}^1$, the points 0 and ∞, which are fixed by the S^1-action, are exceptional. If D contains fixed points of the circle action, then geometric properties of the metric, such as completeness or smooth extendibility, must be studied separately there. Until further notice, it is assumed that fixed points in D (if any) have been removed. The domain D on which the metric lives is therefore a subset of the punctured complex line $\mathbf{C}^\times$. The isothermal coordinates (t, θ) are hereafter identified with the global complex coordinate $w = \exp(t + i\theta)$.

To avoid fixed points, consider an *open* interval $I = (\alpha, \beta)$. A *momentum profile* is a positive function $\varphi\colon I \to \mathbf{R}$, of class C^2 on a neighborhood of the closure. Given a momentum profile φ, the aim is to construct a surface of revolution $\Sigma = (D, g_\varphi)$ and a function $\tau\colon D \to I$ such that

(i) Each level set of τ is an orbit of the circle action.

(ii) The area of the zone $\{\tau_0 \le \tau \le \tau_1\}$ is $2\pi(\tau_1 - \tau_0)$ for all $\tau_0 < \tau_1$ in I.

(iii) The length of the orbit $\{\tau = \tau_0\}$ is $2\pi\sqrt{\varphi(\tau_0)}$ for $\alpha < \tau_0 < \beta$.

Begin by fixing $\tau_0 \in I$ arbitrarily and setting

$$a := \int_{\tau_0}^{\alpha} \frac{dx}{\varphi(x)}, \qquad b := \int_{\tau_0}^{\beta} \frac{dx}{\varphi(x)}. \tag{2.5}$$

Because $1/\varphi > 0$ on (α, β), the equation

$$t = \int_{\tau_0}^{\tau(t)} \frac{dx}{\varphi(x)} \tag{2.6}$$

defines an increasing, differentiable function $\tau\colon (a, b) \to (\alpha, \beta)$. The metric and area form, which *a priori* depend on τ_0, are defined by

$$g_\varphi = \varphi(\tau)\,(dt^2 + d\theta^2), \qquad dA = \varphi(\tau)\,dt \wedge d\theta$$

on the annulus $D = \{e^{t+i\theta} \in \mathbf{C}^\times \mid a < t < b\}$.

Differentiating (2.6) with respect to t, $\tau' = \varphi(\tau)$, so $dA = d\tau \wedge d\theta$. Properties (i)–(iii) follow immediately. The function $s\colon (a, b) \to \mathbf{R}$ defined by

$$s(t) = \int_{\tau_0}^{\tau(t)} \frac{dx}{\sqrt{\varphi(x)}} \tag{2.7}$$

gives the geodesic distance along a generator of Σ by (2.4).

To see analytically that the isometry class of g_φ does not depend on τ_0, introduce the function

$$u(t) = \int_{\tau_0}^{\tau(t)} \frac{x\,dx}{\varphi(x)}. \tag{2.8}$$

Because $\tau' = \varphi(\tau)$, successive differentiation gives

$$u' = \tau, \qquad u''(t) = \tau' = \varphi(\tau),$$

or $u''(t) = e^{\psi(t)}$ in the notation of (2.1). Varying τ_0 changes u by an additive constant, which has no effect on $e^{\psi(t)} = u''(t)$. As a function of t, τ is the inverse of a definite integral; changing the lower limit of integration in (2.5) causes the interval (a,b) to be translated, which does not alter the conformal type of the annulus D. Geometrically, a choice of τ_0 fixes the orbit O in Figure 2.1.

Similar considerations show that the metric associated to the "translated" profile $\tau \mapsto \varphi(\tau - \tau_0)$ is isometric to g_φ for every $\tau_0 \in \mathbf{R}$, as is the metric associated to the "reflected" profile $\tau \mapsto \varphi(-\tau)$. Specifically, a translational change of variable in (2.8) changes u by an added affine function of t, while reflecting reverses the orientation of the t axis. Neither affects the isometry class of the resulting metric. It is therefore harmless to assume, as convenient, that $0 \in I = (\alpha, \beta)$ or (if $I \neq \mathbf{R}$) that $\alpha = 0$.

THE GAUSSIAN CURVATURE

In isothermal coordinates (x,y), the Gaussian curvature of $g = e^{\psi}(dx^2+dy^2)$ is given by the well-known formula

$$K = -\tfrac{1}{2}e^{-\psi}(\psi_{xx} + \psi_{yy}).$$

On a surface of revolution, the conformal factor ψ is independent of θ, so the Gaussian curvature simplifies to

$$K = -\tfrac{1}{2}e^{-\psi(t)}\psi''(t). \tag{2.9}$$

To compute K in terms of τ, first note that the equation $d\tau = \varphi(\tau)\,dt$ implies $e^{-\psi(t)}\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial t} = \varphi(\tau)\frac{\partial}{\partial \tau}$ as vector fields on D. Since $\psi(t) = \log\varphi(\tau)$, substituting in (2.9) gives

$$K = -\frac{1}{2}\frac{\partial}{\partial \tau}\left(\varphi\frac{\partial}{\partial \tau}(\log\varphi)\right) = -\tfrac{1}{2}\varphi''(\tau). \tag{2.10}$$

This striking formula is perhaps the greatest advantage of "momentum" coordinates over more familiar coordinates used in elementary differential geometry.

REMARK 2.1. It is a pleasant, instructive exercise to write out the Laplace-Beltrami operator of g_φ in terms of φ. The resulting formula facilitates the explicit study of spectral geometry, see for example [3] and [7].

COMPLETENESS AND EXTENDIBILITY

Let $\varphi\colon (\alpha,\beta) \to \mathbf{R}$ be a momentum profile, and assume g_φ is defined on a dense subset of a smooth, complete surface of revolution. Each end of the momentum interval corresponds to a topological end of Σ. The "virtual" level set $\{\tau = \beta\}$ corresponds to the orbit $\{t = b\}$ in $\mathbf{P}^1$. If β is finite and

$\varphi(\beta) > 0$, then the metric extends to a metric with a larger *open* momentum interval, so if $\beta < \infty$ then $\varphi(\beta) = 0$. Geometrically this means the orbits must "close up" at an end of finite area, though the virtual orbit may be at finite or infinite distance. If the orbit $\{\tau = \beta\}$ *is* at finite distance, it must be a fixed point (not a circle), so $b = \infty$. Similar remarks hold for the virtual level set $\{\tau = \alpha\}$.

If $\beta = \infty$, then the metric is complete at the corresponding end if and only if

$$\int_{\tau_0}^{\infty} \frac{dx}{\sqrt{\varphi(x)}}$$

diverges. For a rational profile φ, the integral diverges if and only if φ grows no faster than quadratically as $\tau \to \infty$. Analogous observations hold if $\alpha = -\infty$.

Suppose β is finite, and that $\varphi(\beta) = 0$ but $\varphi'(\beta) \neq 0$. Equation (2.6) implies that t is unbounded near $\{\tau = \beta\}$, which means the end contains a fixed point of the circle action[1]). Assume without loss of generality that $\beta = 0$, and consider the zone $\{-\varepsilon \leq \tau \leq 0\} \subset D$, whose boundary has length $2\pi\sqrt{\varphi(-\varepsilon)}$. Let s be the geodesic distance from the fixed point to the boundary. The *cone angle* ϕ at the fixed point is defined to be

$$\phi = \lim_{\varepsilon \to 0+} \frac{2\pi\sqrt{\varphi(-\varepsilon)}}{s},$$

and the metric extends smoothly if and only if $\phi = 2\pi$. L'Hôpital's rule gives

$$\phi = -\varphi'(\beta)\pi, \tag{2.11}$$

so the metric is smooth if and only if $\varphi'(\beta) = -2$. By symmetry, if $-\infty < \alpha$ and the end $\{\tau = \alpha\}$ is at finite distance, then the metric extends smoothly to the fixed point if and only if $\varphi'(\alpha) = 2$.

In the remaining case, φ and φ' both vanish at β. By assumption, the profile has a C^2 extension to a neighborhood of β, so Taylor's theorem implies

$$\varphi(\tau) = \frac{\varphi''(\beta)}{2}(\tau - \beta)^2 + o(\tau - \beta)^2$$

near β. This in turn implies that the arc length integral diverges near β, so the end is complete.

The respective possibilities, with $\beta = \infty$ or 1, are depicted in Figure 2.2.

---

[1]) Alternatively, (2.7) implies the distance to the end is finite, so the end is a puncture as noted previously.

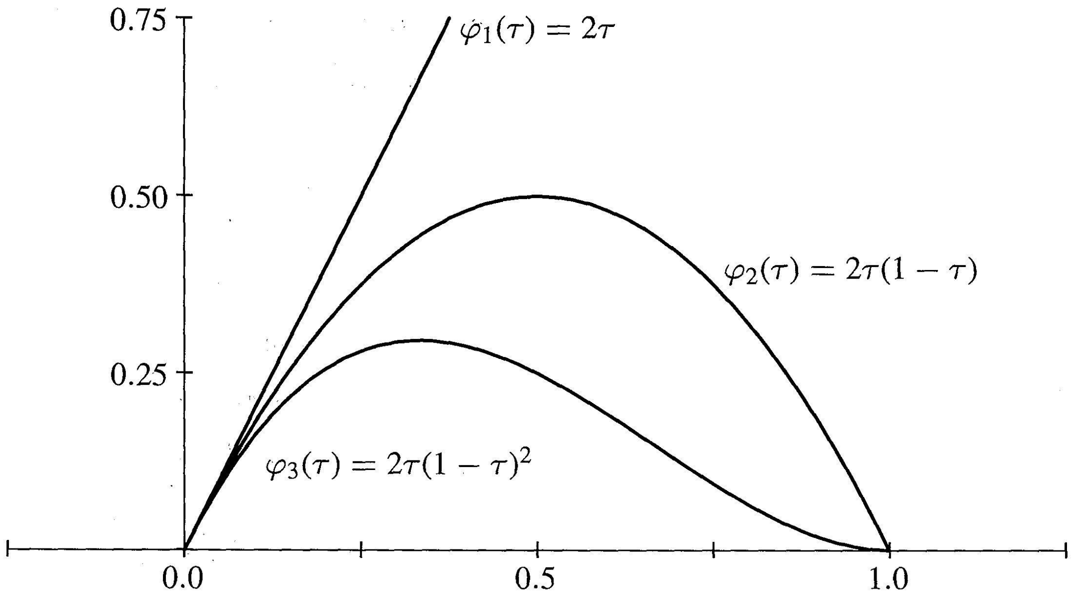

FIGURE 2.2
Momentum profiles inducing smooth, complete metrics

CONICAL SINGULARITIES AND THE GAUSS-BONNET THEOREM. At a smooth point of Σ, the *curvature form* is $K\,dA$, the Gaussian curvature times the area form. If v is an isolated conical singularity, the *angular defect* at v is $2\pi - \phi$, and the curvature form at v is defined to be the angular defect times the δ-function supported at v. Equation (2.11) yields the following observation.

PROPOSITION 2.2. *If $\varphi = 0$ and $\varphi' \neq 0$ at a finite endpoint of the momentum interval, then the angular defect at the corresponding fixed point of Σ is $\big(2 - |\varphi'|\big)\pi$.*

In action-angle coordinates, the Gauss-Bonnet theorem for compact surfaces of revolution is the fundamental theorem of calculus. After scaling and translating we may assume the momentum interval is $[-\beta, \beta]$. By (2.10) and Proposition 2.2, the total curvature of Σ is

$$\int_\Sigma K\,dA = \big(2 + \varphi'(\beta)\big)\pi + \big(2 - \varphi'(-\beta)\big)\pi + 2\pi \int_{-\beta}^{\beta} -\tfrac{1}{2}\varphi'' = 4\pi\,.$$

The Kazdan-Warner integrability condition [5] has a similar interpretation: The Gaussian curvature $K = \kappa(\tau)$ is the second derivative of a function that vanishes at $\tau = \pm\beta$.

CLASSICAL SURFACES OF REVOLUTION

Not every abstract surface of revolution embeds in $\mathbf{R}^3$, even if the image is not assumed to be rotationally symmetric: A famous theorem of Hilbert asserts that the hyperbolic plane cannot even be immersed isometrically in $\mathbf{R}^3$. There is, however, an elementary criterion for embedability, assuming the image is a classical surface of revolution:

PROPOSITION 2.3. *Let Σ be the abstract surface of revolution associated to a momentum profile φ. A portion of Σ embeds in $\mathbf{R}^3$ as a surface of revolution if and only if $|\varphi'| \le 2$ on the corresponding part of the momentum interval.*

Proof. Let ξ be a positive function, and let Σ be the abstract surface obtained by revolving the graph of ξ about the x-axis in $\mathbf{R}^3$. The profile gives the length squared of $\frac{\partial}{\partial\theta}$, namely $\varphi(\tau) = \xi(x)^2$. Differentiating with respect to x,

$$\varphi'(\tau) \cdot \tau'(x) = 2\xi(x) \cdot \xi'(x).$$

Equating the area elements in the classical and momentum descriptions,

$$d\tau = \xi(x)\sqrt{1 + \xi'(x)^2}\, dx.$$

Combining these observations,

$$\varphi'(\tau) = \frac{2\xi'(x)}{\sqrt{1 + \xi'(x)^2}}, \qquad \text{or} \qquad \xi'(x) = \frac{\varphi'(\tau)}{\sqrt{4 - \varphi'(\tau)^2}}.$$

This implies $|\varphi'(\tau)| \le 2$, with equality if and only if $|\xi'(x)| = \infty$.

Several examples are depicted in Section 3.

SUMMARY

For the reader's convenience, here is a concise account of the momentum construction for surfaces of revolution.

DEFINITION 2.4. Let $I \subset \mathbf{R}$ be an interval, possibly unbounded. A *momentum profile* is a function of class C^2 on a neighborhood of the closure of I that is positive on the interior of I.

THEOREM 2.5. *Let $\varphi\colon I \to \mathbf{R}$ be a momentum profile. There exists an abstract surface of revolution (D, g_φ), unique up to isometry, such that*

- *The image of the moment map $\tau\colon D \to \mathbf{R}$ is I.*
- *The orbit $\{\tau = \tau_0\}$ has length $2\pi\sqrt{\varphi(\tau_0)}$ for all $\tau_0 \in I$.*

The Gaussian curvature of g_φ is $K = -\frac{1}{2}\varphi''(\tau)$ wherever the metric is smooth, and the angular defect at a fixed point is $(2 - |\varphi'|)\pi$. The metric is complete at an end $\{\tau = \beta\}$ if and only if one of the following holds:

(INFINITE-AREA END) $|\beta| = \infty$ *and* $\displaystyle\int_{\tau_0}^{\beta} \frac{dx}{\sqrt{\varphi(x)}}$ *diverges.*

(SMOOTH EXTENSION) β *is finite,* $\varphi(\beta) = 0$, *and* $|\varphi'(\beta)| = 2$.

(FINITE-AREA END) β *is finite,* $\varphi(\beta) = 0$, *and* $\varphi'(\beta) = 0$.

3. METRICS OF SPECIFIED CURVATURE

In momentum coordinates, specifying the Gaussian curvature of a metric in terms of zonal area is a matter of integrating twice. The construction is therefore well-adapted to exhibiting a variety of interesting metrics.

CONSTANT CURVATURE

Theorem 2.5 and Proposition 2.3 give a simple classification of surfaces of revolution that have constant Gaussian curvature, together with an easy characterization of when the abstract surface embeds in $\mathbf{R}^3$ as a surface of revolution. Many surfaces of constant negative curvature (such as the pseudosphere) are seen to be portions of *complete* abstract surfaces of revolution.

SMOOTH, COMPLETE METRICS. A metric of constant Gaussian curvature corresponds to a quadratic profile φ, and the metric is smooth and complete if and only if

- $\varphi \geq 0$ on $\mathbf{R}$, or
- $|\varphi'(\beta)| = 2$ at some (hence each) root of φ.

Table 3.1 lists smooth, complete surfaces of revolution that have constant Gaussian curvature. Most of these metrics embed only partially in $\mathbf{R}^3$ as surfaces of revolution, and *no* zone of the Poincaré metric (on the disk Δ) embeds as a surface of revolution. The pseudosphere is the zone in the punctured disk $\Delta^\times$ corresponding to the momentum interval $(0, 1/c^2)$. In the last column, the annulus is determined up to conformal equivalence by the ratio of the inner and outer radii. Each metric is scaled to have curvature $\pm c^2$ or 0, metrics are grouped by the sign of their curvature, and the momentum profiles are translated to have $\alpha = 0$ when possible. For each integrand in Table 3.1, the integrals in equation (2.6) are elementary, and the t integrals can be inverted explicitly.

TABLE 3.1
Smooth, complete, constant-curvature surfaces of revolution

	$\mathbf{P}^1$	$\mathbf{C}$	$\mathbf{C}^\times$	Δ	$\Delta^\times$	Annulus
$\varphi(\tau)$	$2\tau - c^2\tau^2$	2τ	λ^2	$2\tau + c^2\tau^2$	$c^2\tau^2$	$\lambda^2 + c^2\tau^2$
$\tau \in$	$[0, 2c^{-2}]$	$[0, \infty)$	$\mathbf{R}$	$[0, \infty)$	$(0, \infty)$	$(-\infty, \infty)$
$t \in$	$[-\infty, \infty]$	$[-\infty, \infty)$	$\mathbf{R}$	$[-\infty, 0)$	$(-\infty, 0)$	$\left(-\frac{\pi}{c\lambda}, \frac{\pi}{c\lambda}\right)$

OTHER EXAMPLES. Every quadratic polynomial that is positive somewhere gives rise to a surface of constant Gaussian curvature via the momentum construction, though with exactly one exception (the round sphere) the resulting metric is singular and/or not embedable in $\mathbf{R}^3$. There is a pleasant correspondence between quadratic profiles and the "standard zoo" presented in elementary differential geometry. Figures 3.1 and 3.2 depict the negative curvature case; positive curvature is similar.

A remarkable family arises from quadratic profiles $\varphi(\tau) = \lambda^2 - \tau^2$ with $\lambda > 1$. At their smooth points these metrics have unit curvature, yet they admit closed geodesics of length $2\pi\lambda > 2\pi$, in seeming contradiction with Myers' theorem. The discrepancy is resolved by (2.11): The conical singularities at the fixed points carry negative curvature. Viewing these examples as surfaces in $\mathbf{R}^3$, the explanation is different: The portion that embeds is not complete.

AREA, DISTANCE, AND ORBIT LENGTH. An abstract surface of revolution is said to have *bounded orbits* if the profile is a bounded function. If a surface has bounded orbits, and if an end of the surface has finite length, then the end also has finite area. Inversely, if the orbits are not bounded at an end of

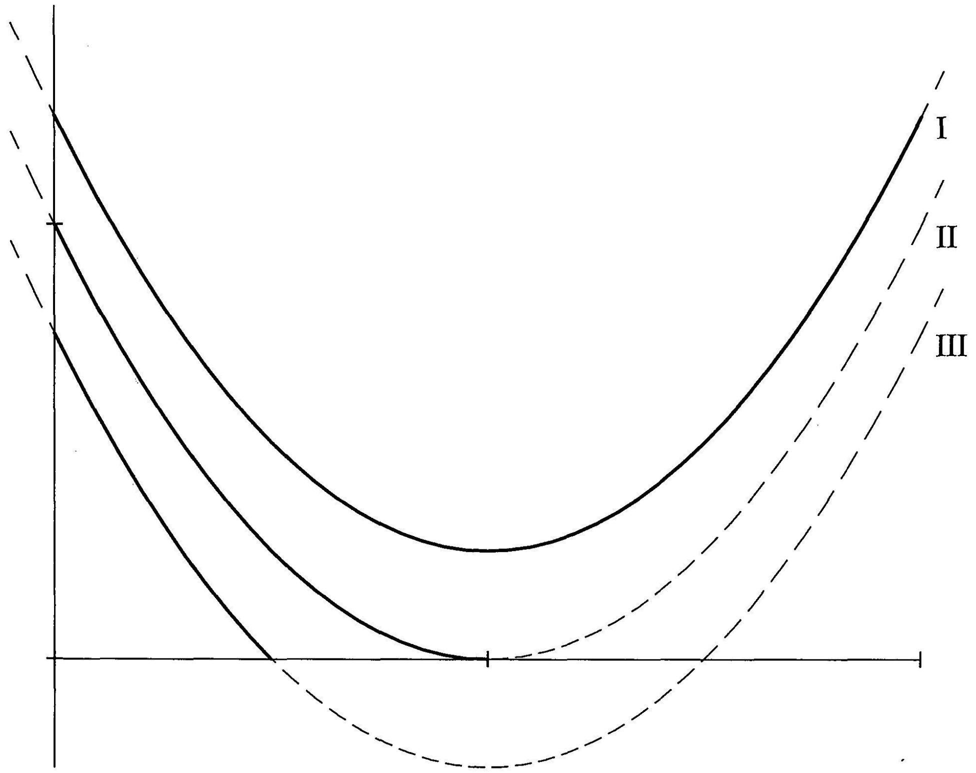

FIGURE 3.1
Quadratic momentum profiles; the heavy portion of each defines a metric that embeds in $\mathbf{R}^3$ as a surface of revolution, see Figure 3.2

infinite length, then the end necessarily has infinite area. Remarkably, these are the only general conclusions that can be drawn about abstract surfaces of revolution; the intuition furnished by surfaces of revolution in $\mathbf{R}^3$ can be misleading! Two examples illustrate what can happen:

- The data $I = [0, \infty)$, $\varphi(\tau) = 2\tau + \tau^3$ define a metric of infinite area on the disk, in which the distance to the edge of the disk is finite.
- The data $I = [0, 1)$, $\varphi(\tau) = 2\tau/(1 - \tau)$ give a metric on the disk with unbounded orbits but having finite area.

Neither metric is complete, and neither can be extended non-trivially.

EXTREMAL METRICS

The *Calabi energy* of a metric g is the integral of the square of the Gaussian curvature,

$$E(g) = \int_\Sigma K^2 \, dA .$$

A metric is *extremal* in the sense of Calabi if the metric is critical for the energy among all smooth metrics of fixed area.

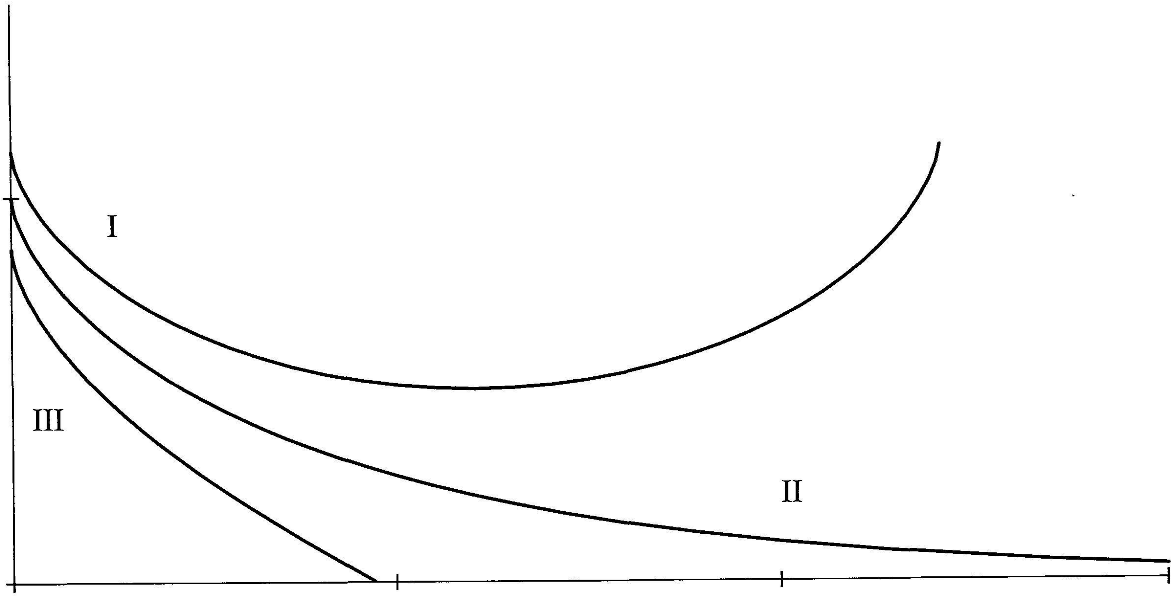

FIGURE 3.2
The geometric profiles that correspond to the momentum profiles in Figure 3.1. Each generates a surface of constant curvature -1

THE CALABI ENERGY FOR METRICS WITH CONICAL SINGULARITIES. The space of surfaces of revolution of area 4π is identified with the space of momentum profiles $\varphi\colon [-1,1] \to \mathbf{R}$. Assume from now on that profiles are of class C^4 and vanish at ± 1. The Gaussian curvature is $K = -\frac{1}{2}\varphi''(\tau)$ for $\tau \in (-1,1)$, while the curvature form is the distribution

$$K\,dA = \pi\Big[\big(2+\varphi'(1)\big)\delta_{(1)} + \big(2-\varphi'(-1)\big)\delta_{(-1)}\Big] - \tfrac{1}{2}\varphi''(\tau)\,dA\,.$$

The Calabi energy is the integral of $K^2\,dA$ over Σ:

$$(3.1)\quad E(g_\varphi) = \frac{\pi}{2}\left[-\varphi''(1)\big(2+\varphi'(1)\big) - \varphi''(-1)\big(2-\varphi'(-1)\big) + \int_{-1}^{1}(\varphi'')^2\right].$$

Differentiating with respect to φ and integrating by parts twice yields

$$(3.2)\quad \frac{2}{\pi}\dot{E}(g_\varphi) = -2\big(\dot{\varphi}''(1)+\dot{\varphi}''(-1)\big) + (\varphi''\dot{\varphi}' - \varphi'\dot{\varphi}'')\Big|_{-1}^{1} + 2\int_{-1}^{1}\varphi^{(4)}\dot{\varphi}\,.$$

For variations supported in $(-1,1)$, the boundary term contributes nothing. Consequently, g_φ is extremal *only if* φ is a cubic polynomial.

REMARK 3.1. The Euler-Lagrange equation (due to Calabi [2]) for a smooth extremal metric on a compact holomorphic manifold is simple and striking: The scalar curvature of the metric is a *holomorphy potential* — a function whose gradient is a holomorphic vector field. To see how this condition is related to φ being cubic, observe first that φ is cubic if and only if the Gaussian curvature (a.k.a. the scalar curvature, since Σ is a complex curve) is an affine function of τ. But the complex gradient of τ is the holomorphic vector field $w\frac{\partial}{\partial w}$. Conversely, affine functions of τ are the *only* S^1-invariant functions with holomorphic gradient, for if $f(\tau)$ is a holomorphy potential on $\mathbf{P}^1$, then $f'(\tau)$ is a global holomorphic function, hence constant.

Calabi [2] showed that the round metric is the only *smooth* extremal metric of area 4π on the sphere. This fact is easily recovered for surfaces of revolution. Smoothness at the ends of the momentum interval means $\varphi'(\pm 1) = \mp 2$. If the profile is not quadratic, there exists $\beta > 1$ such that $\varphi(\tau) = c(1-\tau^2)(\beta - \tau)$. This implies $|\varphi'(-1)| \neq |\varphi'(1)|$, so the metric is not smooth.

Equation (3.2) contains an additional condition for extremality,

$$-2\big(\dot{\varphi}''(1) + \dot{\varphi}''(-1)\big) + (\varphi''\dot{\varphi}' - \varphi'\dot{\varphi}'')\Big|_{-1}^{1} = 0 \quad \text{for all } \dot{\varphi}, \tag{3.3}$$

which says that the energy supported at the fixed points is constant infinitesimally, a condition on the domain of the energy functional as much as a restriction on φ. Without some constraint on the space of metrics, (3.3) is not satisfied, *even for the round metric*. Indeed, the family of profiles $\varphi(\tau) = c(1-\tau^2)$, with $c > 0$, determines a family of metrics for which the curvature concentrates at the fixed points as $c \to 0$, and the energy does not achieve its infimum.

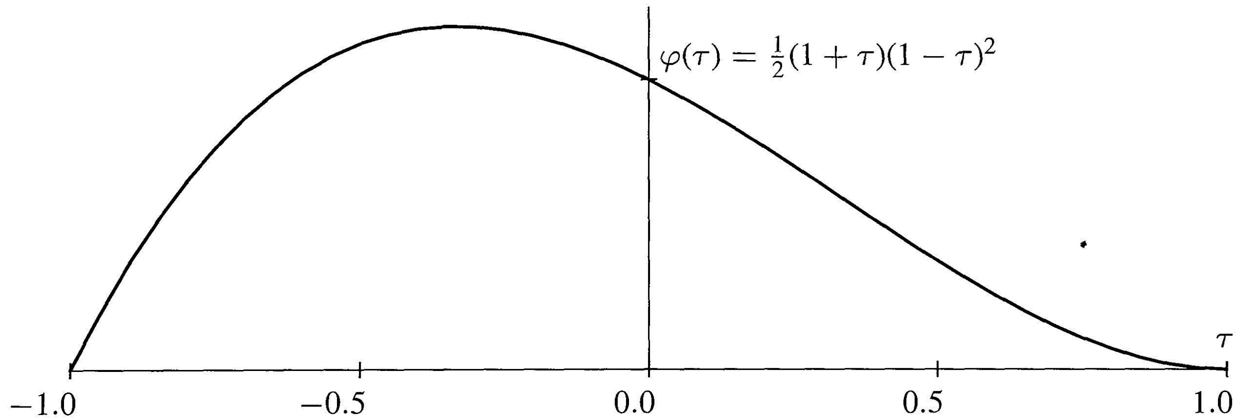

FIGURE 3.3
A cubic momentum profile defining a smooth metric

Two natural constraints on the variation are:

- The energy carried by each end is fixed, so (3.3) holds by fiat.
- The cone angles are fixed, i.e., $\dot{\varphi}'$ vanishes at the endpoints.

When the energy carried by each end is fixed, every metric of constant curvature is critical. In addition, every cubic polynomial

$$\varphi(\tau) = c(1+\tau)(1-\tau)(\beta-\tau)$$

with $\beta \geq 1$ and $c > 0$ gives rise to a critical metric. Among these is a unique *smooth* metric, corresponding to the profile $\varphi(\tau) = \frac{1}{2}(1+\tau)(1-\tau)^2$. The metric g_φ is defined on $\mathbf{C}$ and has area 4π. Because $|\varphi'(\tau)| \leq 2$ for $-1 \leq \tau < 1$ (with equality if and only if $\tau = -1$), the surface $(\mathbf{C}, g_\varphi)$ embeds isometrically in $\mathbf{R}^3$, as a "teardrop" of radius $\frac{4}{3\sqrt{3}}$ and with an infinitely long tail, Figure 3.4.

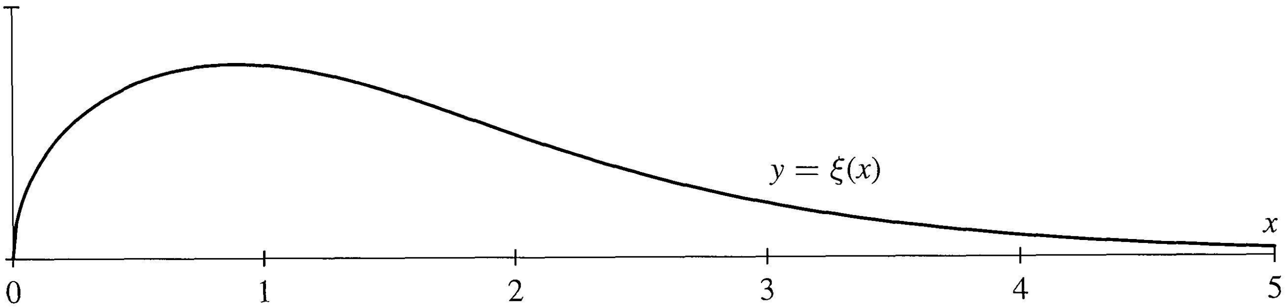

FIGURE 3.4
The embedded geometric profile

Under the weaker restriction that the variation fixes cone angles, the round metric is critical, but is the *only* such metric. Indeed, (3.3) becomes

$$-2\big(\dot\varphi''(1) + \dot\varphi''(-1)\big) - (\varphi'\dot\varphi'')\Big|_{-1}^{1} = 0 \quad \text{for all } \dot\varphi\,.$$

Since φ is cubic, there exist constants a_i such that $\varphi'(\tau) = a_1 + 2a_2\tau + 3a_3\tau^2$. The metric closes up at both ends, so $\int_{-1}^{1} \varphi' = 0$, or $a_1 + a_3 = 0$. A short calculation shows that

$$(a_3 + a_2 + 1)\dot\varphi''(1) + (-a_3 + a_2 + 1)\dot\varphi''(-1) = 0 \quad \text{for all } \dot\varphi\,.$$

Consequently, $a_3 = a_2 + 1 = 0$; this means φ is a quadratic polynomial with leading coefficient -1 that vanishes at ± 1, so g_φ is the round metric.

HISTORY AND ACKNOWLEDGEMENTS

Though the momentum construction arises naturally in symplectic geometry, the author first encountered versions of it in the differential geometry literature. An instance of the integral transform (2.6) appears in a remark of Calabi [1]. The construction as treated in this note perhaps owes its biggest debt to a paper of Koiso and Sakane [6], in which momentum coordinates are used to construct positive Einstein-Kähler metrics. The paper [4] is in part an attempt to frame various differential-geometric constructions in "momentum" language, while simultaneously unifying and generalizing existing results. The momentum construction for surfaces of revolution is elementary, but seems not to be widely appreciated. It is hoped that the present note will help popularize this little gem of differential geometry.

It is a pleasure to thank Michael A. Singer and John Bland for many illuminating discussions, and the referees for several invaluable suggestions.

REFERENCES

[1] CALABI, E. Métriques kählériennes et fibrés holomorphes. *Ann. Sci. École Norm. Sup. (4) 12* (1979), 268–294.

[2] —— Extremal Kähler metrics. In: *Seminar on Differential Geometry* (ed. S.T. Yau), 259–290. *Ann. of Math. Stud. 102*, Princeton Univ. Press, 1982.

[3] ENGMAN, M. Trace formulae for S^1 invariant Green's operators on S^2. *Manuscripta Math. 93* (1997), 357–368.

[4] HWANG, A.D. and M.A. SINGER. A momentum construction for circle-invariant Kähler metrics. *Trans. Amer. Math. Soc. 354* (2002), 2285–2325.

[5] KAZDAN, J. and F.W. WARNER. Curvature functions for compact 2-manifolds. *Ann. of Math. (2) 99* (1974), 14–47.

[6] KOISO, N. and Y. SAKANE. Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds. In: *Curvature and Topology of Riemannian Manifolds*, 165–179. Lecture Notes in Mathematics *1201*, Springer, 1986.

[7] TAIMANOV, I.A. Surfaces of revolution in terms of solitons. *Ann. Global Anal. Geom. 15* (1997), 419–435.

(Reçu le 14 juin 2001; version révisée reçue le 9 janvier 2003)

Andrew D. Hwang

Dept. of Mathematics and Computer Science
College of the Holy Cross
Worcester, MA 01610-2395
U.S.A.
e-mail: ahwang@mathcs.holycross.edu