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QUADRICS, ORTHOGONAL ACTIONS AND INVOLUTIONS IN COMPLEX PROJECTIVE SPACES

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0. INTRODUCTION

The purpose of this article is to look at the canonical action of the special orthogonal group $\mathrm{SO}(n+1,\mathbf{R})$ on $P^n_{\mathbf{C}}$, the complex projective space, in order to get a better understanding of the geometry and topology of the latter. This is related with a classical problem, studied by Zariski [Za] and others, of studying the topology of the complement of an affine algebraic hypersurface $V \subset \mathbf{C}^{n+1}$, in the particular case when V is a homogeneous quadric with an isolated singularity at the origin. We actually look at the projectivized situation. We begin by showing that the complement of a non-singular hyperquadric Q in $P^n_{\mathbf{C}}$ is diffeomorphic to the total space of the tangent bundle of the real projective n-space $P^n_{\mathbf{R}}$,

$$P^n_{\mathbf{C}} \setminus Q \cong T(P^n_{\mathbf{R}})\,.$$

For $n = 3$, this implies that the complement of the nonsingular quadric in $P^3_{\mathbf{C}}$ is diffeomorphic to the group $P\,\mathrm{SL}(2,\mathbf{C})$. Then we use the above observation on the topology of $P^n_{\mathbf{C}} \setminus Q$ to describe $P^n_{\mathbf{C}}$ as the double mapping cylinder of the double fibration

$$\begin{array}{ccccc} & & F^{n+1}_{+}(2,1) & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ Q & & & & P^n_{\mathbf{R}} \end{array}$$

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where

$$F_+^{n+1}(2,1) \cong \mathrm{SO}(n+1,\mathbf{R})/(\mathrm{SO}(n-1,\mathbf{R}) \times \mathbf{Z}/2\mathbf{Z})$$

is the partial flag manifold of *oriented* 2-planes in $\mathbf{R}^{n+1}$ and *non-oriented* lines in these planes. The manifold $F_+^{n+1}(2,1)$ is diffeomorphic to the unit sphere normal bundle of Q in $P^n_{\mathbf{C}}$, and it is also diffeomorphic to the unit sphere tangent bundle of $P^n_{\mathbf{R}}$.

V. Vassiliev pointed out to us that this decomposition of $P^n_{\mathbf{C}}$ resembles the one he gave in [Va1]. In fact, as we explain at the end of Section 3 below, in the case $n = 2$ the above decomposition of $P^2_{\mathbf{C}}$ descends (by complex conjugation) to a similar decomposition of the 4-sphere. If we denote by $F^3(2,1) \cong F^3_+(2,1) / (\mathbf{Z}/2\mathbf{Z})$ the flag manifold of non-oriented 2-planes in $\mathbf{R}^3$ and non-oriented lines in these planes, then the sphere S^4 is obtained by taking the cylinder $F^3(2,1) \times [0,1]$ and gluing two copies of $P^2_{\mathbf{R}}$ to its boundary, via the obvious projections. This is explained (in different words) in Theorem 2 of [Va1], where he uses it to show that the flag manifold $F^3(2,1)$ is the Spanier-Whitehead dual (in the sense of [SW]) of two disjoint copies of $P^2_{\mathbf{R}}$. Also, in the remark at the end of that article, Vassiliev acknowledges an explanation by S. M. Finashin relating to $P^2_{\mathbf{C}}$ his construction on S^4. Thus, in the case $n = 2$ our construction is actually hidden in Vassiliev's article. He explained to us that his method can also be used in higher dimensions to obtain our construction for $P^n_{\mathbf{C}}$ in general. It is interesting to observe, as pointed out to us by E. Ghys, that this decomposition of S^4 actually corresponds to the Tits building for the symmetric space $\mathrm{SL}(3,\mathbf{R})/\,\mathrm{SO}(3,\mathbf{R})$. This quotient can be regarded as the 5-ball, whose *visual sphere* at infinity is S^4. We refer to [BGS, Eb] for details on this construction (especially §9 in Appendix 5 of [BGS]). Our construction for $n = 2$ is also related with the study done by C. T. C. Wall in [Wa] about Klein's formula for real projective plane curves.

In Section 2 we look more carefully at the decomposition of $P^n_{\mathbf{C}}$ arising from the above double fibration. This describes $P^n_{\mathbf{C}}$ as a 1-parameter family of codimension 1 submanifolds $F_+^{n+1}(2,1) \times \{t\}$, for $t \in (0,1)$, together with two "special" fibres: Q and a copy of the real projective space. We prove that these are the orbits of the natural action of $\mathrm{SO}(n+1,\mathbf{R})$ on $P^n_{\mathbf{C}}$, regarded as a subgroup of the complex orthogonal group $\mathrm{SO}(n+1,\mathbf{C})$. This is an isometric action, with respect to the Fubini-Study metric on $P^n_{\mathbf{C}}$, and the principal orbits are the flag manifolds $F_+^{n+1}(2,1)$, which have codimension 1. So this is an isometric action on $P^n_{\mathbf{C}}$ of cohomogeneity 1, thus it is hyperpolar, i.e. there is an embedded geodesic circle, transversal to all the orbits, by [HPTT]. Here we exhibit such a circle explicitly and we use it to parametrize the space of orbits, which is the interval $[0, \frac{\pi}{2}]$. The endpoints of this interval correspond to the two special orbits, which are the quadric Q and the real projective space Π which is the fixed point set of the complex conjugation in $P^n_{\mathbf{C}}$.

For this, it is convenient to look at two other interesting foliations which arise naturally from the double fibration (1.4), and from other considerations too. The first foliation $\mathcal{F}_1$ is defined on $P^n_{\mathbf{C}} \setminus \Pi$ and its leaves are open 2-disks transversal to Q and transversal to all the $\mathrm{SO}(n+1, \mathbf{R})$-orbits on $P^n_{\mathbf{C}} \setminus \Pi$. To construct this foliation, we let $\mathcal{N}$ be the normal map of Q. This map is defined on the normal bundle of Q in $P^n_{\mathbf{C}}$ and it is the restriction to this normal bundle of the exponential map. We show that this map is regular for normal vectors of length less than $\frac{\pi}{2}$ and it carries each normal sphere bundle of Q of radius less than $\frac{\pi}{2}$ into an $\mathrm{SO}(n+1, \mathbf{R})$-orbit. The image under $\mathcal{N}$ of the 2-disks orthogonal to Q are the leaves of the foliation $\mathcal{F}_1$ on $P^n_{\mathbf{C}} \setminus \Pi$. The space Π is the set of *focal points* of Q, i.e. the image under the exponential map of the set of critical values of the normal map. The closure of each leaf of $\mathcal{F}_1$ is a closed 2-disk that meets Π orthogonally in a projective line which is a closed geodesic in $P^n_{\mathbf{C}}$. For each pair of conjugate points in Q, the corresponding leaves are naturally glued together along their common limit set in Π, forming a complex projective line defined by real coefficients. The second foliation $\mathcal{F}_2$ is defined on $P^n_{\mathbf{C}} \setminus Q$; its leaves are embedded n-disks orthogonal to Π. These are the image under the normal map $\mathcal{N}$ of the fibres of the normal disk bundle of Π of radius less than $\frac{\pi}{2}$. The leaves are everywhere transversal to the orbits of $\mathrm{SO}(n+1, \mathbf{R})$. The quadric Q is the set of focal points of Π, and the closure of each leaf in $\mathcal{F}_2$ is a closed n-disk that meets Q orthogonally in a $(n-1)$-sphere, invariant under complex conjugation. The space Π is embedded in $P^n_{\mathbf{C}}$ so that its normal bundle is isomorphic to its tangent bundle, and the leaves of $\mathcal{F}_2$ correspond to the tangent planes of Π, up to isotopy.

As a consequence of these constructions we get that each $\mathrm{SO}(n+1, \mathbf{R})$-orbit in $P^n_{\mathbf{C}}$ is at constant distance from both Q and Π. That is, they are the level sets of the functions "distance to Q" and "distance to Π". The squares of these functions are Bott-Morse functions on $P^n_{\mathbf{C}}$, whose critical set is $Q \cup \Pi$, a result in the spirit of [DR].

In Section 3 we look at the (now classical) theorem saying that $P^2_{\mathbf{C}}$ modulo conjugation is the sphere S^4. This theorem has a long and remarkable history. As explained by V. I. Arnold in [Ar4], he was informed by Rokhlin that this result was known to Pontryagin in the 1930s. The first time this result appeared in print was in 1971, in [Ar1; p. 175], where Arnold used it to study real algebraic curves in $P^2_{\mathbf{R}}$. It was explained to us by Professor Arnold that at the time, it appeared to him that this was an obvious fact which had to be well known, so he stated it without proof. To his surprise, he found no mention of this theorem in the literature, so he asked a number of experts whether they knew about it. In 1973–74 there appeared two independent proofs of this theorem $S^4 \cong P^2_{\mathbf{C}}/conjugation$, given by W. Massey and N. Kuiper [Ku, Ma1]. So we call it the Arnold-Kuiper-Massey theorem (sometimes called the Kuiper-Massey theorem in the literature). Several other proofs of this result have been given by various authors afterwards, including important improvements and generalizations (see for instance [Mar, Mo, Va1, Va2]). We refer particularly to [Ar2], where Arnold gives his original proof, providing a real algebraic map $P^2_{\mathbf{C}} \to S^4$ that induces a diffeomorphism $P^2_{\mathbf{C}}/conjugation \cong S^4$, and to [Ar3, Ar4], where he gives several interesting generalizations following the same method. Different proofs, also with very remarkable generalizations, were given recently by M.F. Atiyah and E. Witten [AW], and by [1]) M.F. Atiyah and J. Berndt [AB].

Here we prove an equivariant version of this theorem, showing that the equivalence $P^2_{\mathbf{C}}/j \cong S^4$ can be realized by a real algebraic map Φ which conjugates the natural cohomogeneity 1 actions of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$ and S^4. Our proof is quite elementary: it uses only linear algebra. The key point is to give appropriate interpretations of $P^2_{\mathbf{C}}$ and S^4. In the case of the sphere, this is given in [HL, DR], where it is observed that S^4 is the set of matrices with norm 1 in the space $\mathcal{S} \cong \mathbf{R}^5$ of real symmetric (3×3) matrices with trace 0. Similarly, $P^2_{\mathbf{C}}$ is the space of complex Hermitian symmetric (3×3) matrices with trace 1 and satisfying $H^2 = H$, i.e. they are orthogonal projections into complex lines (a fact which is well known to the physicists since these lines correspond to states in quantum physics). The map Φ is the one that carries a matrix $H \in P^2_{\mathbf{C}}$ into the unit vector $\psi(H)/||\psi(H)|| \in \mathcal{S}$, where $\psi(H)$ is $[\frac{1}{3}I - \Re(H)]$, I is the (3×3) identity matrix and $\Re$ denotes the real part.

Finally, in Section 4 we use the results and constructions of the previous sections to construct interesting isometric orthogonal actions on $P^3_{\mathbf{C}}$ and S^7, as well as interesting Bott-Morse functions on such manifolds. For this we use the twistor fibration $P^3_{\mathbf{C}} \to S^4$ of Calabi-Penrose, that we describe below, and the beautiful geometry of the quaternions. We also describe the complement of $P^2_{\mathbf{R}}$ in $P^4_{\mathbf{R}}$, embedded as the image of the classical Veronese embedding $P^2_{\mathbf{R}} \hookrightarrow S^4$, followed by the canonical projection of S^4 onto $P^4_{\mathbf{R}}$.

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[1]) We thank Professor Atiyah for explaining to us that our proof is essentially the same as the one in [AB]. This extends to the quaternionic and Cayley planes, which provides corresponding theorems.

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1. ON THE TOPOLOGY OF A QUADRIC IN $P^n_{\mathbf{C}}$

Let Q be a codimension 1, non-singular complex quadric in the projective space $P^n_{\mathbf{C}}$.

THEOREM 1.1. *The complement of Q in $P^n_{\mathbf{C}}$ is diffeomorphic to the total space of the tangent bundle of the n-dimensional real projective space:*

$$P^n_{\mathbf{C}} \setminus Q \cong T(P^n_{\mathbf{R}}) .$$

Proof. We first notice that a non-singular hypersurface of degree d in $P^n_{\mathbf{C}}$ is determined by a homogeneous polynomial of degree d in $n+1$ complex variables, with no critical points outside $0 \in \mathbf{C}^{n+1}$. Let $\mathcal{P}$ be the projective space of coefficients of homogeneous polynomials of degree d in $n+1$ complex variables. The general homogeneous equation of degree d in $n+1$ variables is

$$\sum_{\alpha_0+\cdots+\alpha_n=d} a_{\alpha_0,\ldots,\alpha_n} z_0^{\alpha_0} \ldots z_n^{\alpha_n} = 0 .$$

This defines a polynomial, and hence a hypersurface $\mathcal{X}$, in $\mathcal{P} \times P^n_{\mathbf{C}}$. The family of projective hypersurfaces of degree d in $P^n_{\mathbf{C}}$ is given by the map

$$\mathcal{E} \colon \mathcal{X} \to \mathcal{P} ,$$

induced by the projection of $\mathcal{P} \times P^n_{\mathbf{C}}$ onto $\mathcal{P}$. In $\mathcal{P}$, the polynomials defining singular hypersurfaces in $P^n_{\mathbf{C}}$ form a closed subvariety of complex codimension one. Hence its complement Ω is connected. Since the map $\mathcal{E}$ is a locally trivial fibration over Ω, by Ehresmann's lemma, one knows that any non-singular hypersurface $X \subset P^n_{\mathbf{C}}$ of degree d is ambient isotopic to the hypersurface defined by the Fermat polynomial $\mathbf{F}^n_d := z_0^d + \cdots + z_n^d$. That is, up to isotopy we can assume that X is the projectivization of the affine variety $V := \{z_0^d + \cdots + z_n^d = 0\}$ after removing the singular point $0 \in V$ (cf. [LC; Lemme 2.2]).

The projective space $P^n_{\mathbf{C}}$ is obtained dividing $\mathbf{C}^{n+1} - \{0\}$ by the $\mathbf{C}^*$-action:

$$g_t(z_0, \ldots, z_n) = (e^{it} z_0, \ldots, e^{it} z_n) , \qquad t \in \mathbf{C}^* = \mathbf{C} \setminus \{0\} .$$

Since V is an invariant set for this $\mathbf{C}^*$-action, it follows that $P^n_{\mathbf{C}} \setminus X$ is the image of $\mathbf{C}^{n+1} \setminus V$. Moreover, $\mathbf{C}^*$ is $S^1 \times \mathbf{R}^+$ and if we divide $\mathbf{C}^{n+1} \setminus \{0\}$ by the $\mathbf{R}^+$-action we get the sphere S^{2n-1}. Thus $P^n_{\mathbf{C}} \setminus X$ is the quotient of $S^{2n-1} \setminus (V \cap S^{2n-1})$ by the corresponding S^1-action. By [Mi2], these S^1-orbits are transversal to the Milnor fibres of the polynomial $\mathbf{F}^n_d(z) = z_0^d + \cdots + z_n^d$, and their action on the fibres is given by the monodromy, which is cyclic of period d. Therefore the Milnor fibre F is a d-fold cyclic cover of $P^n_{\mathbf{C}} \setminus X$.

In the quadratic case $d = 2$, the Milnor fibre is diffeomorphic to the affine variety $z_0^2 + \cdots + z_n^2 = 1$. Let us decompose each vector $Z := (z_0, \ldots, z_n)$ into its real and imaginary parts, $Z = U + iV$; then the Milnor fibre is given as the set $(U, V) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ such that $|U|^2 - |V|^2 = 1$ and $U \perp V$. We notice that the map $(U, V) \mapsto (U/\|U\|, V)$ induces an isomorphism of this Milnor fibre with the tangent bundle of S^n. The monodromy is given by multiplication by -1, $(U, V) \mapsto (-U, -V)$. The quotient of F by this involution is, therefore, the tangent bundle of the real projective n-space. □

We notice that part of the argument above is similar to that of Lemmas 2.2 and 2.3 in [LC] (see also Libgober in [Li; Lemma 1.1]), implying Corollary 1.2 below. We denote by X_0 the projectivization of the affine hypersurface defined by the Fermat polynomial $\mathbf{F}^n_d$, and by $C^n_d := P^n_{\mathbf{C}} \setminus X_0$ the complement of X_0.

COROLLARY 1.2. *Let X be a non-singular hypersurface of $P^n_{\mathbf{C}}$ of degree d. Then:*

i) *the pair $(P^n_{\mathbf{C}}, X)$ is isotopic to the pair $(P^n_{\mathbf{C}}, X_0)$; and*

ii) *the Milnor fibre F of $\mathbf{F}^n_d$ is a d-fold cyclic cover of C^n_d, the projection map $F \to C^n_d$ being given by the monodromy of the Milnor fibration of $\mathbf{F}^n_d$ (which is cyclic of period d).*

Since the Milnor fibre has the homotopy type of a bouquet of μ spheres S^n, by [Ph, Mi2], one has (as in [Li]) that for $n > 1$, the fundamental group $\pi_1(C^n_d)$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$, and $\pi_j(C^n_d) \cong \pi_j(\bigvee_\mu S^n)$, for $j > 1$, where $\mu = (d-1)^{n+1}$ is the Milnor number and $\bigvee_\mu S^n$ is a bouquet of μ spheres of dimension n. In particular:

$$(1.3) \qquad \pi_j(C^n_d) = 0 \text{ if } 1 < j < n\,, \quad \text{and} \quad \pi_j(C^n_d) \cong \mathbf{Z}^\mu \text{ if } j = n\,.$$

We now let $Q = Q_{n-1} \subset P^n_{\mathbf{C}}$ be the non-singular hyperquadric in $P^n_{\mathbf{C}}$ with equation

$$z_0^2 + \cdots + z_n^2 = 0\,,$$

in homogeneous projective coordinates. Let $j\colon P^n_{\mathbf{C}} \to P^n_{\mathbf{C}}$ be the involution on $P^n_{\mathbf{C}}$ given by complex conjugation: $j([z_0, \dots, z_n]) = [\bar{z}_0, \dots, \bar{z}_n]$, and let Π be the fixed point set of j, so that $\Pi \cong P^n_{\mathbf{R}}$.

Theorem 1.1 says that $P^n_{\mathbf{C}} \setminus Q$ is diffeomorphic to the tangent bundle $T(\Pi)$, and Π is the zero section of this bundle. Hence $P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$ can be regarded as the set of non zero tangent vectors of Π, so it is diffeomorphic to the cylinder $T_1(\Pi) \times (0,1)$, where $T_1(\Pi)$ is the unit tangent bundle of Q. The group $\mathrm{SO}(n+1, \mathbf{R})$ acts linearly on $\mathbf{C}^{n+1}$ and this action descends to an action on $P^n_{\mathbf{C}}$ which preserves Q. This action also leaves invariant the real projective space Π, where it acts in the usual way (i.e. via the action induced from the linear $\mathrm{SO}(n+1, \mathbf{R})$-action on $\mathbf{R}^{n+1}$). This extends, via the differential, to a transitive action of $\mathrm{SO}(n+1, \mathbf{R})$ on $T_1(\Pi)$, with isotropy subgroup $\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}$. Hence $T_1(\Pi)$ is diffeomorphic to $\mathrm{SO}(n+1, \mathbf{R})/(\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z})$. But $\mathrm{SO}(n+1, \mathbf{R})$ also acts transitively on $F^{n+1}_{+}(2,1)$, the (partial) flag manifold of *oriented* 2-planes in $\mathbf{R}^{n+1}$ and (non-oriented) lines in these planes, with isotropy $\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}$. Thus one has diffeomorphisms

$$T_1(\Pi) \cong \mathrm{SO}(n+1, \mathbf{R})/(\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}) \cong F^{n+1}_{+}(2,1).$$

The Milnor fibre of the Fermat quadric $\mathbf{F}^n_2 = 0$ in $\mathbf{C}^{n+1}$ is diffeomorphic to the total space of the tangent bundle TS^n. Thus the link K of this singularity is diffeomorphic to the unit tangent bundle of S^n. Hence K is diffeomorphic to the Stiefel manifold $V_{n+1,2}$ of orthonormal 2-frames in $\mathbf{R}^{n+1}$. Therefore $Q \subset P^n_{\mathbf{C}}$, being the projectivization of K, is diffeomorphic to the Grassmannian $G_{n+1,2}$ of oriented 2-planes in $\mathbf{R}^{n+1}$. Thus one has a double fibration:

$$\begin{array}{ccc} & F^{n+1}_{+}(2,1) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ Q & & P^n_{\mathbf{R}} \end{array} \tag{1.4}$$

where π_1 and π_2 are the maps that assign to each flag (P, l) either the 2-plane $P \in G_{n+1,2}$ or the line $l \in P^n_{\mathbf{R}}$.

We form the corresponding double mapping cylinder $\big(F^{n+1}_{+}(2,1) \times [0,1]\big)/\sim$, where $\sim$ identifies a point

$$((P_0, l_0), 0) \in F^{n+1}_{+}(2,1) \times \{0\}$$

with the point $\pi_1(P_0, l_0) = P_0$ in $G_{n+1,2} \cong Q$, and a point

$$((P_1, l_1), 1) \in F^{n+1}_{+}(2,1) \times \{1\}$$

with the point $\pi_2(P_1, l_1) = l_1 \in P^n_{\mathbf{R}}$. The space we obtain is homeomorphic to $P^n_{\mathbf{C}}$. Furthermore, the double fibration (1.4) splits into two fibrations, corresponding to the maps π_1 and π_2. In the first case the space we get is the open mapping cylinder of π_1, and this is $P^n_{\mathbf{C}} \setminus \Pi$, while in the second case we get $P^n_{\mathbf{C}} \setminus Q$, which is the open mapping cylinder of π_2. One has the following

THEOREM 1.5. *The projective space $P^n_{\mathbf{C}}$ is the double mapping cylinder of the double fibration (1.4). If we remove Q from $P^n_{\mathbf{C}}$ we obtain a manifold diffeomorphic to the total space of the normal bundle of $\Pi \cong P^n_{\mathbf{R}}$ in $P^n_{\mathbf{C}}$. Reciprocally, if we remove Π from $P^n_{\mathbf{C}}$, what we get is diffeomorphic to the total space of the normal bundle of Q in $P^n_{\mathbf{C}}$. If we remove $Q \cup \Pi$ from $P^n_{\mathbf{C}}$, what we get is diffeomorphic to $F^{n+1}_{+}(2,1) \times (0,1)$, where*

$$F^{n+1}_{+}(2,1) \cong \mathrm{SO}(n+1,\mathbf{R})/\big(\mathrm{SO}(n-1,\mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}\big)$$

is the (partial) flag manifold of oriented 2-planes in $\mathbf{R}^{n+1}$ and (non-oriented) lines in these planes.

Proof. We notice that if we replace in Theorem (1.5) the word *diffeomorphic* by *homeomorphic*, then this theorem follows immediately from the previous discussion. Let us prove that we actually have diffeomorphisms. By Theorem 1.1, this is clear for $P^n_{\mathbf{C}} \setminus Q$. In fact, the fibration of $P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$ given by the manifolds $F^{n+1}_{+}(2,1)$ corresponds to the fibration on $T(\Pi) \setminus \Pi$ given by sphere bundles of radius $r > 0$, for some metric on $T(\Pi)$. These correspond to boundaries of tubular neighbourhoods $\tilde{\nu}_r(\Pi)$ of $\Pi \subset P^n_{\mathbf{C}}$. In particular $P^n_{\mathbf{C}} \setminus Q$ is a tubular neighbourhood of Π, hence $P^n_{\mathbf{C}} \setminus Q$ is diffeomorphic to the total space of the normal bundle of $\Pi \cong P^n_{\mathbf{R}}$ in $P^n_{\mathbf{C}}$. This bundle is isomorphic to $T(\Pi)$.

Let us prove that $P^n_{\mathbf{C}} \setminus \Pi$ is diffeomorphic to the total space of the normal bundle of Q in $P^n_{\mathbf{C}}$. We observe that for all $r > 0$, the interior of $P^n_{\mathbf{C}} \setminus \tilde{\nu}_r(\Pi)$ is diffeomorphic to $P^n_{\mathbf{C}} \setminus \Pi$. Now we prove that $P^n_{\mathbf{C}} \setminus \Pi$ is actually a tubular neighbourhood of Q. For this we recall that if N is a Riemannian submanifold of $P^n_{\mathbf{C}}$, its *normal map* $\mathcal{N}_N$ is the function that associates, to each normal vector v of N in $P^n_{\mathbf{C}}$, the projection to $P^n_{\mathbf{C}}$ (via the exponential map) of the end-point of $v \in TP^n_{\mathbf{C}}$ (see, for instance [Mil], p. 32, or [AG]). Let us denote by $\nu(Q)$ the normal bundle of Q in $P^n_{\mathbf{C}}$ and consider the normal map

$$\mathcal{N}_Q \colon \nu(Q) \to P^n_{\mathbf{C}}\,.$$

We notice that every complex projective line $\mathcal{L}$ in $P^n_{\mathbf{C}}$ orthogonal to Q, for the Fubini-Study metric, is invariant under conjugation, which is an isometry. So $\mathcal{L}$ is defined by equations with real coefficients (cf. §2 below), and it is totally geodesic in $P^n_{\mathbf{C}}$, since it is a complex projective line. Therefore $\mathcal{L}$ intersects Π transversally in a real projective line. This implies that the normal map $\mathcal{N}_Q$ is a diffeomorphism from the open disk bundle in $\nu(Q)$ of radius $\frac{\pi}{2}$ into $P^n_{\mathbf{C}} \setminus \Pi$. The union of all closed geodesic segments normal to Q of length $\frac{\pi}{2}$ fill up all of $P^n_{\mathbf{C}}$. Thus the distance from a point $p \in P^n_{\mathbf{C}} \setminus (\Pi \cup Q)$ to Q is exactly the length of the unique geodesic segment joining p and the unique point $q \in Q$ such that this segment is orthogonal to Q. Hence every tubular neighbourhood of Q in $P^n_{\mathbf{C}}$, of diameter less than $\frac{\pi}{2}$, is diffeomorphic to $P^n_{\mathbf{C}} \setminus \Pi$. □

We remark that one has a construction for the Milnor fibre F of the Fermat polynomial $\mathbf{F}^n_2$ in the spirit of Theorem (1.5), since F can be regarded as the open mapping cylinder of the fibration

$$V_{n+1,2} \cong \mathrm{SO}(n+1,\mathbf{R})/\,\mathrm{SO}(n-1,\mathbf{R}) \longrightarrow \mathrm{SO}(n+1,\mathbf{R})/\,\mathrm{SO}(n,\mathbf{R}) \cong S^n\,,$$

where $V_{n+1,2}$ is the aforementioned Stiefel manifold.

2. On the geometry of $P^n_{\mathbf{C}}$

We now look more carefully at the decomposition of $P^n_{\mathbf{C}}$ arising from the double fibration (1.4). For this, it is convenient to look at two other interesting foliations that arise naturally from the double fibration (1.4), and from other considerations too.

The first foliation $\mathcal{F}_1$ is actually defined on $P^n_{\mathbf{C}} \setminus \Pi$ and its leaves are the fibres of π_1, which are 2-disks transversal to Q, by Theorem 1.5. By construction, each leaf of $\mathcal{F}_1$ is transversal to all the manifolds $F^{n+1}_+(2,1) \times t \subset P^n_{\mathbf{C}}$ for $t \in (0,1)$, intersecting each in a copy of $P^1_{\mathbf{R}}$ and approaching Π as $t \to 1$. Let us construct this foliation in a different way. We endow $P^n_{\mathbf{C}}$ with the Fubini-Study metric. From the proof of Theorem 1.5 we know that the normal map $\mathcal{N}_Q$ of Q induces a diffeomorphism between the open disk bundle of radius $\pi/2$ and $P^n_{\mathbf{C}} \setminus \Pi$. The leaves of $\mathcal{F}_1$ are the images of the normal disks. Since the conjugation $j\colon P^n_{\mathbf{C}} \to P^n_{\mathbf{C}}$ is an isometry, we have that a projective line $\mathcal{L}$ in $P^n_{\mathbf{C}}$ intersects Q at two conjugate points iff it is orthogonal to Q, and this happens iff $\mathcal{L}$ can be defined by equations with real coefficients. So we call these **CR**-lines. If two distinct **CR**-lines intersect, they do so in a point in $\Pi \cong P^n_{\mathbf{R}}$. Also, each **CR**-line $\mathcal{L}$ meets Π in a real projective line, which is an equator of $\mathcal{L}$. Since all complex lines in $P^n_{\mathbf{C}}$ are totally geodesic, the real projective line $\mathcal{L} \cap \Pi$ is a geodesic in $P^n_{\mathbf{C}}$, at equal distance $\pi/2$ from both intersection points in $\mathcal{L} \cap Q$. This divides $\mathcal{L}$ into two round disks of maximal diameter, orthogonal to Q. One can prove that through each point in $P^n_{\mathbf{C}} \setminus \Pi$ there passes a unique **CR**-line, hence these lines foliate this space. Therefore the open disks into which the **CR**-lines split fill out the whole of $P^n_{\mathbf{C}} \setminus \Pi$, they are totally geodesic in $P^n_{\mathbf{C}}$ and orthogonal to Q, thus providing a fibre bundle decomposition of $P^n_{\mathbf{C}} \setminus \Pi$, equivalent to the open disk bundle of the normal bundle $\nu(Q)$ of Q in $P^n_{\mathbf{C}}$. By construction, the closure of each leaf in $P^n_{\mathbf{C}}$ is obtained by attaching to the leaf a real projective line $P^1_{\mathbf{R}} \subset \Pi$, which is its boundary (or limit set). This circle (a real projective line in Π) is invariant by conjugation and it is an equator of a unique **CR**-line, therefore it is also a closed geodesic for the Fubini-Study metric of $P^n_{\mathbf{C}}$.

In the case of the foliation $\mathcal{F}_2$, the leaves are the fibres of π_2, up to isotopy. They are transverse to $F^{n+1}_{+}(2,1) \times t$, for every $t \in (0,1)$, and these leaves are also transverse to Π. We can describe this foliation more precisely as follows. Given $z \in \Pi$, we let $\mathcal{P}_z$ be the pencil of real projective lines in Π passing through z. Note that the tangent vectors at z to the lines of this pencil give the tangent space of Π at z. Let l_z be one of the lines of the pencil $\mathcal{P}_z$. Its complexification is a projective line L_z in $P^n_{\mathbf{C}}$ defined by an equation with real coefficients, invariant under conjugation. This implies that L_z intersects Q at two points w_1 and w_2, which are conjugate; the intersection $L_z \cap Q$ is necessarily orthogonal and l_z is an equator in L_z. Thus, there is a segment $\hat{l}_z$, half of a real projective line (a circle) in L_z, joining the points w_1, z and w_2. This line is orthogonal to Π and to Q, it is geodesic in $P^n_{\mathbf{C}}$ and has length π, by the minimality of L_z. Doing this for all lines in the pencil $\mathcal{P}_z$, we get an open n-disk of radius $\pi/2$ in $P^n_{\mathbf{C}}$, orthogonal to Π at z, filled by geodesics in $P^n_{\mathbf{C}}$ of length $\pi/2$ and intersecting Q orthogonally. Thus the normal map $\mathcal{N}_\Pi$ is regular for vectors of norm $< \pi/2$. The leaves of $\mathcal{F}_2$ are the images under $\mathcal{N}_\Pi$ of the fibres of the open disk normal bundle of $\Pi \subset P^n_{\mathbf{C}}$ of radius $\frac{\pi}{2}$.

There is another interesting way of thinking about this foliation, up to isotopy, which helps to understand the way in which its leaves approach Q. By Corollary 1.2 we have that $P^n_{\mathbf{C}} \setminus Q$ is the Milnor fibre $F := \left\{ z_0^2 + \cdots + z_n^2 = 1 \right\}$ divided by the monodromy $(z_1, \ldots, z_n) \mapsto (-z_1, \ldots, -z_n)$. The fibre F is the tangent bundle of the n-sphere, so it has a natural foliation by leaves diffeomorphic to n-planes. These planes can be described as follows. Let us decompose each $Z := (z_1, \ldots, z_n)$ into its real and imaginary parts, $Z = U + iV$. The fibre F is the set $(U, V) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ such that $||U|| \geq 1$, $||U||^2 - ||V||^2 = 1$ and $U \perp V$. If $||U|| = 1$, then we are on the n-sphere and $||V|| = 0$. Given a fixed $U_0 \in S^n \subset \mathbf{R}^{n+1}$, its "tangent space" is the plane defined as follows: for each $\lambda \in \mathbf{R}$ with $\lambda > 1$, let $S_\lambda(U_0)$ be the

$(n-1)$-sphere in the affine n-plane perpendicular to λU_0, consisting of all vectors V such that the vector $Z = \lambda U_0 + iV$ is in F; these must satisfy $\|V\|^2 = \lambda^2 - 1$. The radius of the sphere $S_\lambda(U_0)$ grows with λ, while for $\lambda = 1$ the corresponding "sphere" is just one point. For a given $U_0 \in S^n$, let us denote by $\mathcal{L}(U_0)$ the union of all these $(n-1)$-spheres $S_\lambda(U_0)$, for all $\lambda \geq 1$. Then $\mathcal{L}(U_0)$ is a copy of $\mathbf{R}^n$ embedded in F as a component of the 2-sheeted hyperboloid consisting of $\mathcal{L}(U_0) \cup \mathcal{L}(-U_0)$. The monodromy map interchanges these two sheets of the hyperboloid, so their image in $P^n_{\mathbf{C}}$ is a manifold diffeomorphic to a plane, that we denote by $\mathcal{F}(U_0)$. By the uniqueness of the tubular neighbourhood, these are the leaves of $\mathcal{F}_2$ up to isotopy.

From this description of $\mathcal{F}_2$ one can see the way in which the leaves approach Q. In fact, let us denote by $S'_\lambda(U_0)$ the image of the sphere $S_\lambda(U_0)$ in $P^n_{\mathbf{C}}$. It lies in $\mathcal{F}(U_0)$. Let $\gamma_\lambda(U_0)$ be the intersection of the unit sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ with the real half cone over $S_\lambda(U_0)$ with vertex at 0. The image of $\gamma_\lambda(U_0)$ in $P^n_{\mathbf{C}}$ is also $S'_\lambda(U_0)$. The sphere $\gamma_\lambda(U_0)$ is the set of vectors $(\frac{\lambda}{\sqrt{2\lambda^2-1}} U_0, \frac{1}{\sqrt{2\lambda^2-1}} V)$ with $(\lambda U_0, V)$ in $S_\lambda(U_0)$. Therefore the limit of $\gamma_\lambda(U_0)$ is the set of vectors $(\frac{1}{\sqrt{2}} U_0, \frac{1}{\sqrt{2}} v)$ where v is $V/\|V\|$, with V as above. Since the vectors $\frac{1}{\sqrt{2}} U_0$ and $\frac{1}{\sqrt{2}} v$ have equal length, the image $\Lambda(U_0)$ in $P^n_{\mathbf{C}}$ of this limit set is in Q, and it is a $(n-1)$-sphere. By continuity, the limit set of $S'_\lambda(U_0)$ in $P^n_{\mathbf{C}}$ is also $\Lambda(U_0)$. Since the conjugate of the vector (U, V) is $(U, -V)$, the sets $\gamma_\lambda(U_0)$ and their limit, are invariant under conjugation. Hence $\Lambda(U_0)$ is also invariant by conjugation.

Let us summarize the previous discussion in the following

PROPOSITION 2.1. *The double fibration (1.4) induces two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ such that:*

i) *The first one $\mathcal{F}_1$ is defined on $P^n_{\mathbf{C}} \setminus \Pi$; its leaves are embedded copies of $\mathbf{R}^2$, orthogonal to Q, which are the images under the normal map of Q of the fibres of the normal disk bundle of Q of radius less than $\frac{\pi}{2}$. The closure of each such leaf is a closed 2-disk that meets Π orthogonally in a projective line which is a closed geodesic in $P^n_{\mathbf{C}}$. For each pair of conjugate points in Q, the corresponding leaves are naturally glued together along their common limit set in Π, forming a complex projective line defined by real coefficients.*

ii) *The second foliation $\mathcal{F}_2$ is defined on $P^n_{\mathbf{C}} \setminus Q$; its leaves are embedded n-disks, orthogonal to Π, which are the images under the normal map of Π of the fibres of the normal disk bundle of Π of radius less than $\frac{\pi}{2}$. The closure of each such leaf is a closed n-disk that meets Q orthogonally in a $(n-1)$-sphere, invariant under complex conjugation.*

We notice that the previous discussion also proves the following fact, that we state as a proposition. We recall that given a Riemannian submanifold N of $P^n_{\mathbf{C}}$, its *focal points* are the critical values of the normal map of N, see [Mi1].

PROPOSITION 2.2. *The real projective space* $\Pi \cong P^n_{\mathbf{R}}$, *consisting of the points in* $P^n_{\mathbf{C}}$ *with homogeneous real coordinates, is the set of focal points of the quadric* Q *defined by the Fermat polynomial* $z_0^2+\cdots+z_n^2=0$. *Conversely, the quadric* Q *is the set of focal points of* Π.

Thus, both manifolds Q and Π can be regarded as *caustics* in $P^n_{\mathbf{C}}$, i.e. they are the critical values of the Lagrangian maps defined by the corresponding co-normal maps of Π and Q, respectively (see [AG]).

Let us consider now the action of $\mathrm{SO}(n+1,\mathbf{R})$ on $P^n_{\mathbf{C}}$, regarded as a subgroup of the complex orthogonal group $O(n+1,\mathbf{C})$. This action leaves Q invariant and it is by isometries with respect to the Fubini-Study metric. An isometry of $P^n_{\mathbf{C}}$ that leaves Q invariant necessarily carries the set of focal points of Q into itself. Hence Π is also an invariant set for the action of $\mathrm{SO}(n+1,\mathbf{R})$. We know already that Q is the Grassmannian $G_{n+1,2} \cong \mathrm{SO}(n+1,\mathbf{R})/(\mathrm{SO}(n-1,\mathbf{R})\times \mathrm{SO}(2,\mathbf{R}))$, so the action of $\mathrm{SO}(n+1,\mathbf{R})$ is transitive on Q. Thus Q is one single orbit, and so is Π. Let us look at the orbit of a point $w \in P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$. We claim that its orbit is the manifold $(F^{n+1}_{+}(2,1) \times t)$ passing through w. For this we use again the normal map

$$\mathcal{N}_Q\colon \nu(Q) \to P^n_{\mathbf{C}}.$$

By the previous discussion, this map is a diffeomorphism from the open disk bundle in $\nu(Q)$ of radius $\frac{\pi}{2}$ into $P^n_{\mathbf{C}} \setminus \Pi$ and the images of the fibres are the leaves of $\mathcal{F}_1$. Hence each point $w \in P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$ is in the image of the normal map $\mathcal{N}_Q$, i.e., there is a (unique) vector $v_w \in \nu(Q)$ normal to Q, such that $w = \mathcal{N}_Q(v_w)$; the norm of v_w equals the distance $d_w = d(w,Q)$ from w to Q, which is > 0 and $< \pi/2$. That is, w corresponds, via $\mathcal{N}_Q$, to a point in the sphere bundle $S_{d_w}(\nu(Q))$ of radius d_w in $\nu(Q)$. We claim that the $\mathrm{SO}(n+1,\mathbf{R})$-orbit $\mathcal{O}_w$ of w is the image of this sphere bundle, i.e. $\mathcal{O}_w = \mathcal{N}_Q(S_{d_w}(\nu(Q)))$. For this we notice that the group $\mathrm{SO}(n+1,\mathbf{R})$ also acts on the tangent bundle $TP^n_{\mathbf{C}}$ via the differential, and this action preserves the (C^∞) splitting $TP^n_{\mathbf{C}}|_Q \cong TQ \oplus \nu(Q)$. This induces an action of $\mathrm{SO}(n+1,\mathbf{R})$ on the normal bundle $\nu(Q)$ of Q, and this action is isometric and commutes with $\mathcal{N}_Q$, proving the claim. Hence the $\mathrm{SO}(n+1,\mathbf{R})$-orbits are all manifolds $(F^{n+1}_{+}(2,1) \times t)$, for some $t \in (0,1)$, with two exceptional orbits which are

Q and Π, corresponding to $t = 0$ and $t = 1$. By [HL; 1.1], this implies that Q and Π are minimal submanifolds of $P^n_{\mathbf{C}}$, which is obvious for Q, being a complex submanifold. The orbits of maximal dimension, which in this case are diffeomorphic to $F^{n+1}_{+}(2,1)$, are called *principal orbits*.

The previous arguments also show that each $\mathrm{SO}(n+1,\mathbf{R})$-orbit in $P^n_{\mathbf{C}}$ is at constant distance from Q, and also from Π, and these distances go from 0 to $\frac{\pi}{2}$. This proves that the space of $\mathrm{SO}(n+1,\mathbf{R})$-orbits in $P^n_{\mathbf{C}}$ is the interval $[0, \frac{\pi}{2}]$, with the two special orbits corresponding to the endpoints of the interval. But one can actually be more precise about this statement. Let us consider again the geodesic $\hat{l}_z$ described above, in the construction of the foliation $\mathcal{F}_2$. In fact we are interested in half of this geodesic segment. To construct this "half geodesic segment", that we shall denote by $\check{l}$, we can start with any complex projective $\mathbf{CR}$-line $\mathcal{L}$. This line intersects Π in a real projective line, and it meets Q orthogonally at two conjugate points, say w and $\overline{w}$. Now we choose a point $z_0 \in \Pi \cap \mathcal{L}$. Then $\check{l}$ is the geodesic (of length $\frac{\pi}{2}$) in $\mathcal{L}$ joining the points z_0 and w, and it is a geodesic in $P^n_{\mathbf{C}}$ because $\mathcal{L}$ is totally geodesic. This geodesic $\check{l}$ starts at $z_0 \in \Pi$ and ends at $w \in Q$. Hence it meets each $\mathrm{SO}(n+1,\mathbf{R})$-orbit orthogonally in exactly one point, since the orbits are the level sets of the function distance to Π. Hence $\check{l}$ parametrizes the orbits of $\mathrm{SO}(n+1,\mathbf{R})$. This shows that the $\mathrm{SO}(n+1,\mathbf{R})$-action on $P^n_{\mathbf{C}}$ is a hyperpolar isometric action of cohomogeneity 1, which is already well known (see for instance [HPTT, Ko]). In fact, *cohomogeneity 1* means that the principal orbits have codimension 1, and we know that this happens in our case. An isometric action is said to be *polar* if there exists a closed, connected submanifold Σ that meets all orbits orthogonally. In our case this can be, for instance, the complete geodesic in $\mathcal{L}$ determined by $\check{l}$. Such a manifold is called a *section*. If one can choose such a section to be also flat, one says that the action is *hyperpolar*. This is obviously satisfied in our case since the section is a geodesic.

We have thus proved the following

THEOREM 2.3.

i) *The natural* $\mathrm{SO}(n+1,\mathbf{R})$*-action on* $P^n_{\mathbf{C}}$ *is an isometric, hyperpolar action of cohomogeneity* 1, *whose space of orbits is the interval* $[0,\pi/2]$. *A section for this action (i.e. a submanifold that intersects transversally each orbit at exactly one point) can be constructed by considering some (any)* $\mathbf{CR}$*-line* $\mathcal{L}$, *choosing a point* $z \in \mathcal{L} \cap \Pi$ *and taking the geodesic (a circle) in* $\mathcal{L}$ *that passes through* z *and the two points where* $\mathcal{L}$ *meets* Q.

ii) *There are three orbit types: two special orbits, Q and Π, which correspond to the endpoints $\{0, \pi/2\}$, and the principal orbits, which are copies of the partial flag manifold,*

$$F_{+}^{n+1}(2,1) \cong \mathrm{SO}(n+1, \mathbf{R})/(\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}),$$

of oriented 2-planes in $\mathbf{R}^{n+1}$ and lines in these planes. The manifold $F_{+}^{n+1}(2,1)$ is diffeomorphic to the unit sphere normal bundle of Q in $P_{\mathbf{C}}^{n}$, and also to the unit sphere tangent bundle of $P_{\mathbf{R}}^{n}$. Each of the two special orbits is the set of focal points of the other, and they are minimal submanifolds of $P_{\mathbf{C}}^{n}$.

iii) *The complex projective lines in $P_{\mathbf{C}}^{n}$ whose homogeneous coordinates are real, i.e. the* **CR***-lines, foliate $P_{\mathbf{C}}^{n} \setminus \Pi$ and they are everywhere transversal to the orbits of* $\mathrm{SO}(n+1, \mathbf{R})$ *(away from Π). In particular, they are orthogonal to Q.*

iv) *The real projective space $\Pi \cong P_{\mathbf{R}}^{n}$ is embedded in $P_{\mathbf{C}}^{n}$ so that its normal bundle is isomorphic to its tangent bundle. Its "tangent spaces" naturally define a foliation of $P_{\mathbf{C}}^{n} \setminus Q$ by embedded copies of $\mathbf{R}^{n}$, which are everywhere transversal to the orbits of* $\mathrm{SO}(n+1, \mathbf{R})$ *(away from Q). In particular, they are orthogonal to Π.*

We now let $q \colon P_{\mathbf{C}}^{n} \to [0, \pi/2] \subset \mathbf{R}$ be the function $q(Z) = [d(Z, Q)]^2$, i.e. q is the square of the distance to Q. It is clear that q is constant along the $\mathrm{SO}(n+1, \mathbf{R})$-orbits, which are its level sets. Hence q has the two special orbits Q and Π as critical set. It is clear that if Σ is a small disk in $P_{\mathbf{C}}^{n}$ orthogonal to Q (or to Π), then the restriction of q to Σ is the ordinary quadratic map, so it is a Morse function on Σ. This means, by definition, that q is a Bott-Morse function. We have thus obtained the following results, motivated by [DR]:

COROLLARY 2.4. *The map q is a Bott-Morse function, whose level surfaces are the orbits of* $\mathrm{SO}(n+1, \mathbf{R})$ *and the critical set consists of the two special orbits Q and $\Pi \cong P_{\mathbf{R}}^{n}$.*

Of course one can replace the function q above by $p(Z) = [d(Z, \Pi)]^2$, which is also a Bott-Morse function.

COROLLARY 2.5. *Let* $\pi\colon S^{2n+1} \to P^n_{\mathbf{C}}$ *be the Hopf fibration. Then the composition* $p \circ \pi\colon S^{2n+1} \to [0, \pi/2] \subset \mathbf{R}$ *is a Bott-Morse function. The critical set has two components, which are* S^1*-bundles over the two special orbits in Theorem 2.2. One of these is the Stiefel manifold* $V_{n+1,2} \subset S^{2n+1}$ *of real oriented* 2*-planes in* $\mathbf{R}^{n+1}$*, diffeomorphic to the link of the affine Fermat quadric, the other is the unique non-trivial* S^1*-bundle over* $P^n_{\mathbf{R}}$*. (Both of these are minimally embedded submanifolds of* S^{2n+1}*, by* [HL].*)*

REMARKS 2.6.

i) We notice that if we let $S^2(P^1_{\mathbf{C}})$ be the symmetric product $(P^1_{\mathbf{C}} \times P^1_{\mathbf{C}})/I$, where I is the involution $I(x,y) = (y,x)$, then there is a canonical holomorphic surjection $p\colon P^1_{\mathbf{C}} \times P^1_{\mathbf{C}} \to S^2(P^1_{\mathbf{C}})$ taking (x,y) to the point $[(x,y)]$ in $S^2(P^1_{\mathbf{C}})$. This induces an isomorphism $S^2(P^1_{\mathbf{C}}) \cong P^2_{\mathbf{C}}$. Hence, every identification $Q \cong P^1_{\mathbf{C}}$ also determines an analytic isomorphism $S^2(Q) \cong P^2_{\mathbf{C}}$, where the conic Q in $P^2_{\mathbf{C}}$ is the image of the diagonal Δ. This is, essentially, a special case of the projective Vieta Theorem, which says that $P^n_{\mathbf{C}}$ is the n^{th} symmetric power of $P^1_{\mathbf{C}}$. A real version of this result was proved by Arnold in [Ar3; Th. 2].

ii) Let us denote by j the antipodal map in $P^1_{\mathbf{C}} \cong \mathbf{C} \cup \{\infty\}$. This is given by $j(z) = -1/\bar{z}$, and is a fixed point free involution of $P^1_{\mathbf{C}}$. The anti-diagonal (the graph of the antipodal map) in $P^1_{\mathbf{C}} \times P^1_{\mathbf{C}}$ is given by

$$\Delta^{-1} := \{(z, -1/\bar{z})\} .$$

This gives a copy of $P^1_{\mathbf{C}}$ anti-holomorphically embedded in $(P^1_{\mathbf{C}} \times P^1_{\mathbf{C}}) \setminus \Delta$. It is clear that Δ^{-1} is invariant under the involution $I(x,y) = (y,x)$ of $(P^1_{\mathbf{C}} \times P^1_{\mathbf{C}})$. Thus $\Delta^{-1} := \{(z, -1/\bar{z})\}$ is projected onto a smooth copy of $P^2_{\mathbf{R}}$ in $P^2_{\mathbf{C}}$, disjoint from Q. Hence the identification ϕ of $P^1_{\mathbf{C}}$ with $Q \subset P^2_{\mathbf{C}}$ also determines, canonically, a copy of the real projective space $P^2_{\mathbf{R}}$ in $P^2_{\mathbf{C}} \setminus Q$, together with an involution of $P^2_{\mathbf{C}}$ whose fixed point set is this $P^2_{\mathbf{R}}$. If Q is the Fermat conic, $\{z_1^2 + z_2^2 + z_3^2 = 0\}$, then this embedding of $P^2_{\mathbf{R}}$ in $P^2_{\mathbf{C}}$ is the usual one.

iii) We notice that, also for $n = 2$, every diffeomorphism $f\colon Q \to Q$ extends canonically to a diffeomorphism $\widetilde{f}\colon P^2_{\mathbf{C}} \to P^2_{\mathbf{C}}$, and this extension is functorial, i.e., $\widetilde{f_2 \circ f_1} = \widetilde{f}_2 \circ \widetilde{f}_1$ (cf. [Gh]). In fact, through every point $u \in P^2_{\mathbf{C}} - Q$, there are two tangents to Q, which determine points $\phi(a_1), \phi(a_2)$ in Q. Then $\widetilde{f}(u)$ is the point of intersection of the lines tangent to Q at the points $f(\phi(a_1))$ and $f(\phi(a_2))$. A consequence of these remarks is that if G is a group acting on Q, then the G-action extends to $P^2_{\mathbf{C}}$. In particular, if G is $\mathrm{SO}(3, \mathbf{R})$, acting on $Q \cong S^2$ by rotations, its extension to $P^2_{\mathbf{C}}$ is the action that we considered in Sections 2 and 3 above. Similarly, if G is $\mathbf{Z}/2\mathbf{Z}$ acting on Q as the antipodal map, then the corresponding extension to $P^2_{\mathbf{C}}$ is given by complex conjugation.

3. $P^2_{\mathbf{C}}$ AND THE 4-SPHERE S^4

The previous discussion, restricted to $n = 2$ and compared to the cohomogeneity 1 isometric action of $\mathrm{SO}(3,\mathbf{R})$ on S^4 constructed in [HL], motivates an equivariant version of the Arnold-Kuiper-Massey theorem [Ar1, Ar2, Ku, Ma1], saying that $P^2_{\mathbf{C}}$ modulo conjugation is the 4-sphere. In this section we give a new proof of this theorem. We construct an explicit algebraic map $\Phi\colon P^2_{\mathbf{C}} \to S^4$, which is equivariant with respect to the cohomogeneity 1 isometric actions of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$ and S^4 and induces a diffeomorphism $P^2_{\mathbf{C}}/conjugation \cong S^4$.

We start by recalling the $\mathrm{SO}(3,\mathbf{R})$-action on S^4, as explained by Hsiang and Lawson in [HL; Example 1.4].

Let $\mathcal{S}$ be the vector space of real 3×3, traceless and symmetric matrices. As a real vector space $\mathcal{S}$ is $\mathbf{R}^5$, and it can be equipped with a metric given by the inner product $(A,B) \mapsto \mathrm{trace}(AB)$. Let $\mathcal{S}^{(4)}$ be the space of matrices in $\mathcal{S}$ with norm 1. One has an obvious diffeomorphism $S^4 \cong \mathcal{S}^{(4)}$, which becomes isometric if we endow S^4 with its usual round metric and $\mathcal{S}^{(4)}$ with the metric given by the inner product in $\mathcal{S}$. We shall identify these two spaces in the sequel, denoting both of them by S^4 indistinctly. The group $\mathrm{SO}(3,\mathbf{R})$ acts on $\mathcal{S}$ by $A \mapsto O^tAO$, where O^t is the transposed matrix (which is equal, in our case, to O^{-1}). This induces an isometric action Γ of $\mathrm{SO}(3,\mathbf{R})$ on S^4. This action on S^4 has two disjoint copies of $P^2_{\mathbf{R}}$ as special fibres (see the remark at the end of this section). The space of orbits is the interval $[0,1]$, with the endpoints giving the special orbits. Each principal orbit (i.e. the orbits of highest dimension) is a flag manifold

$$F^3(2,1) \cong \mathrm{SO}(3,\mathbf{R})\,/\,(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \cong L(4,1)\,/\,(\mathbf{Z}/2\mathbf{Z}),$$

of pairs (P,l) with P a plane in $\mathbf{R}^3$ and l line in P, where $L(4,1)$ is the lens space $S^3\,/\,(\mathbf{Z}/4\mathbf{Z}) \cong \mathrm{SO}(3,\mathbf{R})\,/\,(\mathbf{Z}/2\mathbf{Z})$.

Let us give a similar description of $P^2_{\mathbf{C}}$. Let

$$\mathfrak{H}(3,\mathbf{C}) = \{H \in M(3,\mathbf{C}) \mid H = H^*\}$$

be the space of complex 3×3 Hermitian matrices, where $H^* = \overline{H}^t$ is the adjoint matrix of H, obtained by first conjugating each entry of H and then transposing the matrix. We equip $\mathfrak{H}(3, \mathbf{C})$ with the Hermitian inner product

$$\langle H_1, H_2 \rangle = \frac{1}{2} \operatorname{trace}(H_1 H_2). \tag{3.1}$$

As a vector space, with this inner product, $\mathfrak{H}(3, \mathbf{C})$ is the ordinary Euclidean space $\mathbf{E}^9$. Consider the subset $P(2)$ of $\mathfrak{H}(3, \mathbf{C})$ defined by

$$P(2) = \left\{ H \in \mathfrak{H}(3, \mathbf{C}) \mid H^2 = H \quad \text{and} \quad \operatorname{trace}(H) = 1 \right\}. \tag{3.2}$$

LEMMA 3.3. *The set $P(2)$ is a manifold, diffeomorphic to $P^2_{\mathbf{C}}$. Moreover, if we endow $P(2)$ with the metric defined by (3.1), then $P(2)$ is isometric to $P^2_{\mathbf{C}}$ equipped with the Fubini-Study metric (of constant holomorphic sectional curvature 4).*

We remark that it is possible to describe $P^n_{\mathbf{C}}$ in a similar way, but we restrict our attention to $n = 2$ because this is all we need.

Proof. We claim that if H is in $P(2)$, then it is an orthogonal projection over a complex line. In fact, if H is in $P(2)$, then it is diagonalizable by a unitary matrix and its eigenvalues are 0 or 1, because $H^2 = H$. Since the trace is one, two eigenvalues must be 0 and the other is 1. Hence H is a surjection of $\mathbf{C}^3$ over a complex line, and this map has to be an orthogonal projection because H is Hermitian. Conversely, it is clear that each line $L \in \mathbf{C}^3$ determines a unique orthogonal projection of $\mathbf{C}^3$, and this is given by a matrix in $P(2)$. The diffeomorphism in Lemma 3.3 is achieved by the map that carries H into the corresponding line in $\mathbf{C}^3$. To prove that this map gives a metric equivalence, we notice that the unitary group $U(3)$ acts on $\mathfrak{H}(3, \mathbf{C})$ by $H \mapsto U^* H U$, and $P(2)$ is an orbit of this action, with isotropy $\big(U(2) \times U(1)\big)$. Thus,

$$P(2) \cong U(3) / \big(U(2) \times U(1)\big) \cong P^2_{\mathbf{C}},$$

and the metric on $P(2)$ is obviously $U(3)$-invariant. Hence the induced metric on $P^2_{\mathbf{C}}$ is also $U(3)$-invariant, and this characterizes the Fubini-Study metric, up to scaling. □

We recall now that the quotient of $P^2_{\mathbf{C}}$ by the complex conjugation j is a smooth manifold, which is not an obvious fact since j has fixed points. This is carefully explained in [Mar], so we only sketch a few ideas here. Away from the fixed point set $\Pi \cong P^2_{\mathbf{R}}$, the involution j is free, so the quotient is a smooth manifold. The problem is on Π. A tubular neighbourhood of

Π in $P^2_{\mathbf{C}}$ can be regarded as an open disk normal bundle, and conjugation carries each normal fibre into itself. Since the quotient of each normal 2-disk by the involution is again a 2-disk, it follows that the quotient $P^2_{\mathbf{C}}/j$ is a topological manifold. Making this argument more carefully one gets that $P^2_{\mathbf{C}}/j$ is in fact a PL-manifold, as noticed in [Ku], and therefore it is smooth, since every piecewise linear 4-manifold is smooth. In [Mar] Marin defines the smooth structure on $P^2_{\mathbf{C}}/j$ directly, without using PL-structures. An important point is that the smooth structure on $P^2_{\mathbf{C}}/j$ is such that the obvious projection $P^2_{\mathbf{C}} \to P^2_{\mathbf{C}}/j$ is differentiable.

Let us denote by Γ the aforementioned isometric action of $\mathrm{SO}(3,\mathbf{R})$ on S^4, and by $\widetilde{\Gamma}$ the standard action of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$, which is by isometries with respect to the Fubini-Study metric. This action is defined either by considering $\mathrm{SO}(3,\mathbf{R})$ as a subgroup of $O(3,\mathbf{C})$, acting on the space of lines in $\mathbf{C}^3$, or via the action of $\mathrm{SO}(3,\mathbf{R})$ on the space of matrices $P(2) \subset H(3,\mathbf{C})$ given by

$$(O,A) \mapsto O^t A O\,.$$

By Lemma 3.3, both metrics on $P^2_{\mathbf{C}}$ are equivalent; also for every $O \in \mathrm{SO}(3,\mathbf{R})$, $H \in P(2)$ and $v \in \mathbf{C}^3$ such that $H(v) = v$, one has $O^t H O(O^{-1}(v)) = O^{-1}(v)$, because $O^{-1} = O^t$. Hence both actions on $P^2_{\mathbf{C}} \cong P(2)$ are equivalent. Similarly, given the $\mathrm{SO}(3,\mathbf{R})$-actions $\widetilde{\Gamma}$ on $P^2_{\mathbf{C}}$ and Γ on S^4, we say that these actions are equivariant if there exists a map $\Phi\colon P^2_{\mathbf{C}} \to S^4$ which makes the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{SO}(3,\mathbf{R}) \times P^2_{\mathbf{C}} & \xrightarrow{\ \widetilde{\Gamma}\ } & P^2_{\mathbf{C}} \\
{\scriptstyle Id\times\Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\mathrm{SO}(3,\mathbf{R}) \times S^4 & \xrightarrow{\ \Gamma\ } & S^4\,.
\end{array}$$

In this case we say that Φ *conjugates* the actions Γ and $\widetilde{\Gamma}$. The map Φ carries orbits into orbits, i.e. the decompositions of $P^2_{\mathbf{C}}$ and S^4 into orbits are (smoothly) equivalent.

Let us now state the equivariant Arnold-Kuiper-Massey theorem:

THEOREM 3.4. *There is a real algebraic equivariant map $\Phi\colon P^2_{\mathbf{C}} \to S^4$, which is invariant by the complex conjugation j and induces a diffeomorphism $P^2_{\mathbf{C}}/j \cong S^4$, providing a conjugation between the isometric* $\mathrm{SO}(3,\mathbf{R})$*-actions $\widetilde{\Gamma}$ on $P^2_{\mathbf{C}}$ and Γ on S^4.*

We notice that Theorem 3.4, together with [HL], imply that the image of $P^2_{\mathbf{R}} \subset P^2_{\mathbf{C}}$ under the above map is the image of $P^2_{\mathbf{R}}$ by the classical Veronese embedding $(P^2_{\mathbf{C}}, P^2_{\mathbf{R}}) \hookrightarrow (P^5_{\mathbf{C}}, S^4)$.

The proof of Theorem 3.4 follows from several lemmas below.

LEMMA 3.5. *Let A be a real (3×3)-matrix. Then A is the real part of a matrix H in $P(2)$ if and only if*

i) *A is symmetric with trace 1;*

ii) *A has 0 as an eigenvalue and the other two eigenvalues λ_i and λ_j are roots of an equation of the form:*

$$\lambda^2 - \lambda + k = 0\,,$$

for some constant $k \in \mathbf{R}$ with $0 \leq k \leq \frac{1}{4}$.

If A and H are as above, and if $O \in \mathrm{SO}(3, \mathbf{R})$ is such that $O^t A O$ is a diagonal matrix, then the imaginary part B of H, taken into its canonical form $O^t B O$, has only two possible non-zero entries, which are $\pm\sqrt{k}$. In particular, if $k = 0$, then $H = A$.

Proof. Let us consider a matrix $H \in P(2)$ and decompose it into its real and imaginary parts: $H = A + iB$. Then one has $\overline{H}^t = A^t - iB^t$. Also $H = \overline{H}^t$ because H is Hermitian. Hence $A = A^t$ and $B = -B^t$, i.e. A is symmetric and B is anti-symmetric. Thus the trace of A is 1, proving statement (i). One also has

$$H^2 = A^2 - B^2 + i(AB + BA)\,,$$

and $H^2 = H$ because H is in $P(2)$. Therefore $A = A^2 - B^2$ and $B = AB + BA$.

Now, A is symmetric, and so is A^2; these two matrices obviously commute, so they can be diagonalized simultaneously by a matrix $O \in \mathrm{SO}(3, \mathbf{R})$. Since $B^2 = A^2 - A$, one knows that $O^t B^2 O$ is also diagonal:

$$O^t B^2 O = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix},$$

with $\mu_i = \lambda_i^2 - \lambda_i$, for each $i = 1, 2, 3$, where the λ_i are the eigenvalues of A. But B is antisymmetric and commutes with B^2, which is symmetric. Hence the same matrix O takes B to its canonical form:

$$O^t B O = \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix}$$

for some $a, b, c \in \mathbf{C}$. This implies that

$$O^tB^2O = (O^tBO)(O^tBO) = \begin{pmatrix} -a^2-c^2 & -bc & ab \\ -bc & -a^2-b^2 & -ac \\ ab & -ac & -b^2-c^2 \end{pmatrix},$$

which we know is a diagonal matrix. Therefore two of the numbers a, b, c must be zero. Assume for instance that a and b are 0, then both eigenvalues λ_1 and λ_3 are roots of the polynomial

$$\lambda^2 - \lambda + c^2 = 0.$$

This implies that

$$\lambda_1 + \lambda_3 = 1 \quad \text{and} \quad \lambda_1 \cdot \lambda_3 = c^2 \geq 0.$$

Hence $\lambda_2 = 0$ (because the trace of A is 1), so 0 is an eigenvalue of A. The other eigenvalues λ_1 and λ_3 must both be ≥ 0 and ≤ 1, because their product is non-negative and their sum is 1. Moreover the roots must be real, therefore $k = c^2 \leq \frac{1}{4}$, proving statement (ii).

Also, in this case the eigenvalues of A determine the imaginary part B of H up to sign:

$$B = \pm O \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} O^t,$$

with $c^2 = \lambda_1 - \lambda_3^2 = \lambda_3 - \lambda_3^2$, proving in this case the last statement of Lemma 3.5. The other cases, when either $a = c = 0$ or $b = c = 0$, are similar to the previous one. This proves that if $A = \Re(H)$ for some matrix $H \in P(2)$, then A is as stated in Lemma 3.5. Conversely, given A satisfying these conditions, the above arguments tell us how to construct B so that these matrices are the real and imaginary parts of some H in $P(2)$. □

Now, given $H \in P(2)$, its real part is $\Re(H) = \frac{1}{2}(H + \overline{H})$. Define

$$\psi \colon P(2) \to M(3, \mathbf{R}),$$

the space $M(3, \mathbf{R})$ being the space of real (3×3)-matrices, by the formula

$$\psi(H) = \frac{1}{3} I_3 - \Re(H) \in M(3, \mathbf{R}), \tag{3.6}$$

where I_3 is the (3×3)-identity matrix. In other words, $\psi(H)$ is the real part of the matrix $(\frac{1}{3}I_3 - H)$. Since $H \in P(2)$, it follows that $\psi(H)$ is actually contained in $\mathcal{S}$.

It is clear that the above action of $\mathrm{SO}(3, \mathbf{R})$ on $P(2)$ given by conjugation is equivalent, via the above diffeomorphism $P(2) \cong P^2_{\mathbf{C}}$, with the standard action studied in §2 and §3 above. It is also clear that, for every $O \in \mathrm{SO}(3,\mathbf{R})$, one has

$$\psi(O^tHO) = \frac{1}{3}I - \frac{1}{2}\big(O^t(H+\overline{H})O\big) = O^t\big(\frac{1}{3}I - \frac{1}{2}(H+\overline{H})\big)O = O^t\psi(H)O\,.$$

Hence we have

LEMMA 3.7. *The map ψ is equivariant. That is, for every $O \in \mathrm{SO}(3,\mathbf{R})$ and $H \in P(2)$, one has $\psi(O^tHO) = O^t\psi(H)O$.*

LEMMA 3.8. *Given $S \in \mathcal{S} - \{0\}$, there exists a unique positive $t \in \mathbf{R}$, such that the matrix $(\frac{1}{3}I - tS)$ is the real part of some matrix $H \in P(2)$.*

Proof. By Lemma 3.7, we may assume that S is diagonal. Hence the matrix $\widehat{S}_t = (\frac{1}{3}I - tS)$ is also diagonal, say

$$\widehat{S}_t = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{pmatrix}$$

with $\lambda_i(t) = \frac{1}{3} - t\mu_i$, where the μ_i are the eigenvalues of S. We notice that for all $t \in \mathbf{R}$, one has

$$\operatorname{trace}\widehat{S}_t = 1 - t\,(\operatorname{trace} S) = 1\,,$$

because S has trace 0. Hence all these matrices satisfy condition (i) of Lemma 3.5.

Let us look for the possible values of t that give solutions of Lemma 3.5. That is, we want $t > 0$ for which one eigenvalue $\lambda_i(t)$ is 0 and the others are such that their sum is 1 and their product is ≥ 0 and $\leq \frac{1}{4}$.

Let us number the eigenvalues of S so that $\mu_1 \leq \mu_2 \leq \mu_3$. Since their sum is 0 and S is not the zero matrix, one must have $\mu_1 < 0$ and $\mu_3 > 0$. If we want t as above, one $\lambda_i(t)$ must vanish. Let us look for solutions with $\lambda_1(t) = 0$. This means that $t = \frac{1}{3\mu_1} < 0$, and we want $t > 0$. Hence, there are no solutions with $\lambda_1(t) = 0$.

Now let us look for solutions with $\lambda_2(t) = 0$. This implies that $t = \frac{1}{3\mu_2}$; for this to be possible we must have $\mu_2 \neq 0$. If $\mu_2 < 0$, then $t < 0$ and we want t to be positive. Thus, we only care about $\mu_2 > 0$. We have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_2}) \quad \text{and} \quad \lambda_3(t) = \frac{1}{3}(1 - \frac{\mu_3}{\mu_2})\,.$$

We have $\mu_1 < 0 < \mu_2$, so $\lambda_1(t) > 0$. If $\mu_2 < \mu_3$, then $\lambda_3(t) < 0$, thus the product $\lambda_1(t)\lambda_3(t)$ is < 0, so there are no such solutions to Lemma 3.8. The other possibility is $\mu_2 = \mu_3$; this also implies $\lambda_3(t) = 0$. In this case one has $\lambda_1(t) = 1$ and $\lambda_2(t) = \lambda_3(t) = 0$, and $t = \frac{1}{3\mu_2}$ is positive. Hence we have a solution, and this is unique because $\mu_2 = \mu_3$. If $\mu_2 = 0$, then $\lambda_2(t)$ cannot be 0 and we cannot find solutions like this.

Summarizing, so far we have seen that: i) there are no solutions as in Lemma 3.8 for which $\lambda_1(t) = 0$; ii) if $\mu_2 \leq 0$, there are no solutions as in Lemma 3.8 for which $\lambda_2(t) = 0$; and iii) if $\mu_2 = \mu_3$, then there is a unique solution as in Lemma 3.8, for which $\lambda_2(t) = \lambda_3(t) = 0$ and $\lambda_1(t) = 1$.

Finally, let us look for solutions with $\lambda_3(t) = 0$, i.e. with $t = \frac{1}{3\mu_3}$. We know, by hypothesis, that $\mu_2 \leq \mu_3$ and $\mu_3 > 0$. If $\mu_2 = \mu_3$, then we are in the previous case and there is a unique positive t giving a solution as in Lemma 3.8. Let us assume now that $\mu_2 < \mu_3$. Then we have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_3}) \quad \text{and} \quad \lambda_2(t) = \frac{1}{3}(1 - \frac{\mu_2}{\mu_3}),$$

which are both ≥ 0. Since their sum is 1, it follows that each $\lambda_i(t)$ is also ≤ 1.

The product of $\lambda_1(t)$ and $\lambda_2(t)$ satisfies

$$\begin{aligned} 0 \leq \lambda_1(t) \cdot \lambda_2(t) &= \frac{1}{9}(1 - \frac{\mu_1 + \mu_2}{\mu_3} + \frac{\mu_1\mu_2}{\mu_3^2}) = \frac{1}{9}(2 + \frac{\mu_1\mu_2}{\mu_3^2}) \\ &= \frac{1}{9}(2 + \frac{\mu_1\mu_2}{(\mu_1 + \mu_2)^2}) \leq \frac{1}{4}, \end{aligned}$$

since $\mu_1 + \mu_2 + \mu_3 = 0$ and $\frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2} \leq \frac{1}{4}$ because $\frac{1}{4}(a+b)^2 \geq ab$ for any real numbers a and b (with equality if and only if $a = b$). Hence $t = \frac{1}{3\mu_3}$ is the unique solution satisfying the conditions of Lemma 3.8. □

We now "normalize" the map ψ so that its image is contained in $S^4 \subset \mathcal{S}$. For this we define a function

$$\alpha(H) = [\text{trace}(\psi(H)^2)]^{-\frac{1}{2}},$$

i.e. $\alpha(H)$ is the inverse of the norm of $\psi(H)$ in $\mathcal{S}$, and we set

$$\Phi(H) = \alpha(H)\,\psi(H).$$

One has

$$\begin{aligned} \text{trace}[\psi(H)^2] &= \text{trace}[(\frac{1}{3}I_3 - \frac{1}{2}(H + \overline{H}))^2] \\ &= \text{trace}[\frac{1}{9}I_3 - \frac{1}{3}(H + \overline{H}) + \frac{1}{4}(H^2 + \overline{H}^2 + H\overline{H} + \overline{H}H)] \\ &= \frac{1}{6} + \frac{1}{4}\text{trace}(H\overline{H} + \overline{H}H), \end{aligned}$$

which is always positive since the matrix $(H\overline{H}+\overline{H}H)$ is positive semi-definite, so its trace is ≥ 0. Hence the maps α and Φ are well defined. It is clear that the image of Φ is contained in $S^4 \subset \mathcal{S}$, because the linearity of the trace implies that

$$[\text{trace}(\Phi(H))]^2 = \alpha^2(H)\,[\text{trace}\,\psi(H)]^2 = 1\,.$$

It is also clear that Φ is $\mathrm{SO}(3,\mathbf{R})$-equivariant, since the trace is invariant under conjugation and ψ is equivariant by Lemma 3.7. These considerations imply both Lemma 3.8 and the following

LEMMA 3.9. *The map Φ is an equivariant surjection from $P(2)$ over $S^4 \subset \mathcal{S}$, and it is two-to-one, except over the image of the real matrices in $P(2)$ where it is one-to-one.*

This gives the map in Theorem 3.4 that determines an equivariant diffeomorphism between S^4 and $P^2_{\mathbf{C}}$ modulo the involution given by conjugation. To complete the proof of Theorem 3.4 we need to show that Φ is invariant under the involution of $P(2)$ that corresponds to complex conjugation in $P^2_{\mathbf{C}}$. For this we notice that if L_H is the complex line in $\mathbf{C}^3$ which is the image of $H \in P(2)$, and if $0 \neq (z_1, z_2, z_3) \in L_H$, we can associate to H the point in $P^2_{\mathbf{C}}$ with projective coordinates $[z_1, z_2, z_3]$. To the matrix $\overline{H}$ there corresponds the line with projective coordinates $[\overline{z}_1, \overline{z}_2, \overline{z}_3]$. Therefore we have

LEMMA 3.10. *The involution $j*$ of $P(2)$ defined by $j*(H) = \overline{H}$ coincides with the involution j of $P^2_{\mathbf{C}}$ given by complex conjugation,* $[z_1, z_2, z_3] \stackrel{j}{\mapsto} [\overline{z}_1, \overline{z}_2, \overline{z}_3]$.

Then Φ is invariant under this involution, since $\Re(H) = \Re(\overline{H})$, proving Theorem 3.4. □

4. SOME APPLICATIONS AND REMARKS

It is interesting to describe explicitly the orbits of the Γ action of $\mathrm{SO}(3,\mathbf{R})$ on S^4, regarded[2]) as the set of matrices with norm 1 in $\mathcal{S}$. In fact, the orbits of this action are conjugacy classes (or *congruency classes*) of traceless symmetric matrices whose square has trace 1. This is the connection between our construction and the spherical Tits buildings. Every $S \in \mathcal{S}$ can

[2]) This orbit description of S^4 is also given in [Ma2].

be diagonalized by an element in $SO(3, \mathbf{R})$, hence every orbit has a unique representative which is diagonal. So let us assume that S is diagonal with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. The two special orbits correspond to the cases when two eigenvalues coincide. Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, if two eigenvalues coincide we must have, up to conjugation, $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{6}}$ and $\lambda_3 = -\frac{2}{\sqrt{6}}$, or $\lambda_1 = \lambda_2 = -\frac{1}{\sqrt{6}}$ and $\lambda_3 = \frac{2}{\sqrt{6}}$. In both cases the corresponding matrix is determined by the plane P given by the two equal eigenvalues, say λ_1 and λ_2. Equivalently, this matrix is determined by the line orthogonal to P, in which we act by the multiplier $\lambda_3 = \pm\frac{2}{\sqrt{6}}$; the sign here distinguishes the two orbits. Since $SO(3, \mathbf{R})$ acts transitively on the lines in $\mathbf{R}^3$, it follows that each of these special orbits is a copy of $P^2_{\mathbf{R}}$, as we know from [HL]. The general orbits occur when the three eigenvalues are distinct and the corresponding eigenspaces are orthogonal lines. Since the trace is 0, two eigenvalues determine the third. Hence in each case the transformation is determined by the plane P given by two eigenvalues and the line l in P given by one of them, together with the corresponding multipliers on l, on the line orthogonal to l in P and on the line orthogonal to P in $\mathbf{R}^3$. That is, we have a flag (P, l) in $\mathbf{R}^3$, together with the multipliers λ_1, λ_2 and λ_3. Since the action of $SO(3, \mathbf{R})$ is transitive on the planes in $\mathbf{R}^3$ and on the lines in each such plane, it follows that each principal orbit, a copy of the flag manifold $F^3(2,1)$, is the orbit of the flag (P, l). The different orbits correspond to the different multipliers.

We also notice that there is a double fibration, similar to the one considered in (1.4) above:

(4.1)
$$\begin{array}{ccc} & F^3(2,1) & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ P^2_{\mathbf{R}} & & P^2_{\mathbf{R}} \end{array} .$$

where $\pi_1(P, l) = l$ and $\pi_2(P, l) = P$. We can form the corresponding double mapping cylinder $\big(F^3(2,1) \times [0,1]\big)/\sim$, where $\sim$ identifies a point $((P_0, l_0), 0) \in \big(F^3(2,1) \times \{0\}\big)$ with the point $\pi_1(P_0, l_0) = l_0 \in P^2_{\mathbf{R}}$, and a point $((P_1, l_1), 1) \in \big(F^3(2,1) \times \{1\}\big)$ with the point $\pi_2(P_1, l_1) = P_1 \in P^2_{\mathbf{R}}$. We obtain S^4.

The double fibration given by (1.4) in this dimension descends to (4.1) by conjugation. By the previous discussion, the image of Q in S^4 is the copy of $P^2_{\mathbf{R}}$ which is the orbit of the diagonal matrix with eigenvalues $\left\{-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\}$, while Π is taken diffeomorphically into the orbit of $\left\{\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right\}$.

Since the action of $SO(3,\mathbf{R})$ in S^4 is by isometries and is transitive on each orbit, the principal orbits are at constant distance from each of these exceptional orbits $M_1 \cong P^2_{\mathbf{R}}$ and $M_2 \cong P^2_{\mathbf{R}}$, i.e. they are "parallel". In other words, as in Section 2, the principal orbits are the level sets of the function $f\colon S^4 \to \mathbf{R}$ given by $f(x) = (d(x,M_1))^2$ (or the level sets of the function $g(x) = (d(x,M_2))^2$). Both f and g are smooth Bott-Morse functions (cf. [DR]).

The fixed-point free involution on S^4 given by $\iota\colon A \in \mathcal{S} \mapsto -A \in \mathcal{S}$ commutes with our $SO(3,\mathbf{R})$ action and therefore it takes $SO(3,\mathbf{R})$-orbits into orbits. The quotient S^4/ι is the real projective space $P^4_{\mathbf{R}}$, equipped with an isometric $SO(3,\mathbf{R})$-action. The two exceptional orbits M_1 and M_2 on S^4 are identified by ι. Thus we have only one exceptional orbit for the action of $SO(3,\mathbf{R})$ on $P^4_{\mathbf{R}}$. The orbit N of the matrix in S^4 which corresponds to the matrix in $\mathcal{S}$ whose eigenvalues are $\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$ is the manifold consisting of points such that $d(x,M_1) = d(x,M_2)$. Then, N is invariant under ι and separates S^4 into two regions which are interchanged by ι (i.e. N is an "equator" for the orientation-reversing involution ι). The orientable 3-manifold N is the flag manifold described earlier, but it can also be described as the set of ordered pairs (l_1, l_2) of *non-oriented* lines of $\mathbf{R}^3$ which are mutually orthogonal. These lines are the eigenspaces corresponding to the eigenvalues $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$, respectively.

The restriction of ι to N is the orientation-preserving and fixed-point free involution given by $(l_1,l_2) \mapsto (l_2,l_1)$. Let π denote the double covering map from S^4 to $P^4_{\mathbf{R}} = S^4/\iota$. Let $\pi(M_1) = \pi(M_2) := M \cong P^2_{\mathbf{R}}$ and $\pi(N) := \widehat{N}$. The manifold $\widehat{N}$ is diffeomorphic to $SO(3,\mathbf{R})/D_4$, where D_4 is the group of order 8 of isometries of the square. This is because $SO(3,\mathbf{R})$ acts transitively on the set of non-oriented pairs $\{l_1,l_2\}$ of lines in $\mathbf{R}^3$ which are mutually orthogonal and the isotropy group is precisely D_4. Therefore $\widehat{N}$ is diffeomorphic to $SU(2)/\widetilde{D}_4 \cong S^3/\widetilde{D}_4$, where $\widetilde{D}_4$ is the binary dihedral group of order 16, i.e. $\widetilde{D}_4 = \phi^{-1}(D_4)$ where $\phi\colon S^3 \cong SU(2) \to SO(3,\mathbf{R})$ is the canonical epimorphism.

The embedding $P^2_{\mathbf{R}} \cong M \subset P^4_{\mathbf{R}}$ is exactly the embedding given by the Veronese embedding $P^2_{\mathbf{R}} \to S^4$, followed by the canonical projection from S^4 into $P^4_{\mathbf{R}}$ (see [HL]). We know that $S^4 \setminus (M_1 \cup M_2)$ is diffeomorphic to $N \times \mathbf{R}$, and the restriction of the involution ι to $S^4 \setminus (M_1 \cup M_2)$ is conjugate to the involution $\mathfrak{I}$ of $N \times \mathbf{R}$ given by $((l_1,l_2),t) \mapsto ((l_2,l_1),-t)$. Therefore the quotient $(N \times \mathbf{R})/\mathfrak{I}$ is diffeomorphic to the total space of the non-orientable line bundle over $\widehat{N}$. Summarizing, we have the following

COROLLARY 4.2. *Let $P^2_{\mathbf{R}} \cong M \subset P^4_{\mathbf{R}}$ be the embedding induced by the classical Veronese embedding $P^2_{\mathbf{R}} \hookrightarrow S^4$. Then $P^4_{\mathbf{R}} \setminus M$ is diffeomorphic to the total space of the non-orientable real line bundle over* $\mathrm{SU}(2)/\widetilde{D}_4 = S^3/\widetilde{D}_4$. *In particular the fundamental group of $P^4_{\mathbf{R}} \setminus M$ is the binary dihedral group of order* 16.

Let us now recall that there is a remarkable fibre bundle $\pi : P^3_{\mathbf{C}} \to S^4$ with fibre $P^1_{\mathbf{C}}$, called *the twistor fibration*, or also the *Calabi-Penrose fibration* (we refer to [Sa, SV] for details). The fibres are called *the twistor lines*. There are several equivalent ways to construct this fibration. The standard way is to think of $P^3_{\mathbf{C}}$ as being the homogeneous space $\mathrm{SO}(5, \mathbf{R})/U(2)$, which fibres over $\mathrm{SO}(5, \mathbf{R})/\mathrm{SO}(4, \mathbf{R}) \cong S^4$ with fibre $\mathrm{SO}(4, \mathbf{R})/U(2) \cong S^4 \cong P^1_{\mathbf{C}}$. A more geometric way of describing this twistor fibration is to consider S^4 as being the quaternionic projective line $P^1_{\mathcal{H}}$, of *right* quaternionic lines in the quaternionic plane $\mathcal{H}^2$ (regarded as a 2-dimensional *right* $\mathcal{H}$-module). That is, for $q := (q_1, q_2) \in \mathcal{H}^2$ $(q \neq (0,0))$, the right quaternionic line passing through q is the linear space

$$R_q := \{(q_1\lambda, q_2\lambda) \mid \lambda \in \mathcal{H}\}\,.$$

We can identify $\mathcal{H}^2$ with $\mathbf{C}^4$ via the $\mathbf{R}$-linear map given by $(q_1, q_2) \mapsto (z_1, z_2, z_3, z_4)$, where $q_1 = z_1 + z_2\mathbf{j} = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ and $q_2 = z_3 + z_4\mathbf{j} = y_1 + y_2\mathbf{i} + y_3\mathbf{j} + y_4\mathbf{k}$. In this notation $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the standard quaternionic units, $z_1 = x_1 + x_2\mathbf{i}$, $z_2 = x_3 + x_4\mathbf{i}$, $z_3 = y_1 + y_2\mathbf{i}$ and $z_4 = y_3 + y_4\mathbf{i}$.

Under this identification each right quaternionic line is invariant under *right* multiplication by $\mathbf{i}$. Hence such a line is canonically isomorphic to $\mathbf{C}^2$. If we think of $P^3_{\mathbf{C}}$ as being the space of complex lines in $\mathbf{C}^4$, then there is an obvious map $\pi : P^3_{\mathbf{C}} \to S^4$, whose fibre over a point $H \in P^1_{\mathcal{H}}$ is the space of complex lines in the given right quaternionic line $H \cong \mathbf{C}^2$; thus the fibre is $P^1_{\mathbf{C}}$.

The group $\mathrm{Conf}_+(S^4)$ of orientation preserving conformal automorphisms of S^4 is isomorphic to $P\mathrm{SL}(2, \mathcal{H})$, the projectivization of the group of 2×2, invertible, quaternionic matrices. This is naturally a subgroup of $P\mathrm{SL}(4, \mathbf{C})$, since every quaternion corresponds to a couple of complex numbers. Hence $\mathrm{Conf}_+(S^4)$ has a canonical lifting to a group of holomorphic transformations of $P^3_{\mathbf{C}}$, carrying twistor lines into twistor lines.

Let us split (differentiably) the tangent bundle of $P^3_{\mathbf{C}}$ into a horizontal sub-bundle and a "vertical" sub-bundle (the bundle tangent to the twistor fibres), via the Levi-Civita connexion of the metric. Since the lifting of $\mathrm{Conf}_+(S^4)$ permutes the twistor lines, this action on $TP^3_{\mathbf{C}}$ preserves the decomposition into horizontal and vertical sub-bundles. By [SV], the action on the vertical sub-bundle is by isometries with respect to the Fubini-Study metric (which is just the standard metric on S^2). We remark that the horizontal bundle is a *holomorphic* complex sub-bundle of rank two of the complex tangent bundle of $P^3_{\mathbf{C}}$. On this sub-bundle, the action is conformal. However, the group $\mathrm{SO}(5,\mathbf{R})$ is a subgroup of $\mathrm{Conf}_+(S^4)$ and, by construction, its induced action on the horizontal sub-bundle is by isometries. Thus we have an isometric action of $\mathrm{SO}(5,\mathbf{R})$ on $P^3_{\mathbf{C}}$, with respect to the Fubini-Study metric, which restricts to an isometric action of $\mathrm{SO}(3,\mathbf{R})$ on $P^3_{\mathbf{C}}$, via the representation Γ of this group in $\mathrm{SO}(5,\mathbf{R})$ discussed earlier. We denote this latter action of $\mathrm{SO}(3,\mathbf{R})$ on $P^3_{\mathbf{C}}$ by $\check{\Gamma}$.

We notice that the special orbits of the $\mathrm{SO}(3,\mathbf{R})$-action on S^4 give rise to the special orbits in $P^3_{\mathbf{C}}$, each being diffeomorphic to $P^2_{\mathbf{R}}$. There is one such orbit for each point in the twistor line over a point in the corresponding special orbit in S^4. Since the twistor bundle is trivial when restricted to any proper subset of S^4, it follows that the set of all special orbits of each type is diffeomorphic to $P^2_{\mathbf{R}} \times P^1_{\mathbf{C}}$. Similar remarks apply to the principal orbits. Moreover, by [HL], each special orbit is a minimal submanifold of $P^3_{\mathbf{C}}$, and so is their product $P^2_{\mathbf{R}} \times P^1_{\mathbf{C}}$ since the projection $P^3_{\mathbf{C}} \to S^4$ is a harmonic map which is a Riemannian fibration (i.e. it is transversally isometric), by [EL] and [EV; 7.9]. Thus we have

THEOREM 4.3. *The action* $\check{\Gamma}$ *of* $\mathrm{SO}(3,\mathbf{R})$ *on* $P^3_{\mathbf{C}}$ *is such that:*

(1) *The action is by elements of* $P\mathrm{SU}(4)$, *i.e. by isometries of* $P^3_{\mathbf{C}}$ *that permute the twistor lines, sending each twistor line isometrically onto its image.*

(2) *There are two exceptional types of orbits, each of which is diffeomorphic to* $P^2_{\mathbf{R}}$. *If we denote by* K_1 *and* K_2 *the union of orbits of each of these two types, then both* K_1 *and* K_2 *are diffeomorphic to* $P^2_{\mathbf{R}} \times P^1_{\mathbf{C}}$. *Furthermore,* K_1 *and* K_2 *are minimally embedded in* $P^3_{\mathbf{C}}$.

(3) *The principal orbits are diffeomorphic to* $F^3(2,1)$. *Hence the action has cohomogeneity* 3.

(4) *The functions* $h_1\colon P^3_{\mathbf{C}} \to \mathbf{R}$ *and* $h_2\colon P^3_{\mathbf{C}} \to \mathbf{R}$, *given by* $h_1(Z) = (d(Z,K_1))^2$ *and* $h_2(Z) = (d(Z,K_2))^2$, *are both Bott-Morse functions with critical set* $K_1 \cup K_2$.

(5) *The space of orbits is* $S^2 \times [0,1]$.

We may now consider the Hopf fibration $\tilde{\pi}\colon S^7 \to S^4$ and we identify $\mathbf{R}^8 \cong \mathcal{H}^2$ in the obvious way.

The group SU(2) consists of all 2×2 complex matrices of the form $\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ with determinant 1. This group can be identified with the group Sp(1) of unit quaternions by mapping each such matrix to the unit quaternion $u = z_1 + z_2\mathbf{j}$. Hence SU(2) acts by the right on $\mathcal{H}^2 \cong \mathbf{R}^8$ by the map $((q_1, q_2), u) \mapsto (q_1 u, q_2 u)$, for each $u \in \mathrm{Sp}(1)$ and $(q_1, q_2) \in \mathcal{H}^2$. This action leaves invariant each right line R_q $(q = (q_1, q_2))$ and it acts as an isometry on this line.

On the other hand, each complex number is a quaternion, so each matrix in SU(2) can be regarded as a 2×2 quaternionic matrix in $\mathrm{GL}(2, \mathcal{H})$, the group of all invertible 2×2 quaternionic matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This group acts on $\mathcal{H}^2$ by the left according to the formula

$$q = (q_1, q_2) \mapsto (aq_1 + bq_2, cq_1 + dq_2) = A(q)\,,$$

and induces the aforementioned action of $P\mathrm{SL}(2, \mathcal{H})$ on $P^1_{\mathcal{H}} \cong S^4$.

We thus have an action of $\mathrm{SU}(2) \times \mathrm{Sp}(1)$ on $\mathbf{R}^8 \cong \mathcal{H}^2$ by the formula

$$((g, u), (q_1, q_2)) \mapsto (a_g q_1 u + b_g q_2 u, c_g q_1 u + d_g q_2 u)$$

for each $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$ in SU(2). This action induces a natural action

$$\widehat{\Gamma}\colon (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times S^7 \to S^7$$

on the sphere S^7, and this action is a lifting of the action Γ considered in Section 3, i.e. the following diagram is commutative:

$$\begin{array}{ccc} (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times S^7 & \xrightarrow{\ \widehat{\Gamma}\ } & S^7 \\ {\scriptstyle f \times \tilde{\pi}}\downarrow & & {\scriptstyle \tilde{\pi}}\downarrow \\ \mathrm{SO}(3, \mathbf{R}) \times S^4 & \xrightarrow{\ \Gamma\ } & S^4\,, \end{array}$$

where $f(g, u) = \phi(g)$, ϕ being the canonical epimorphism from SU(2) to $\mathrm{SO}(3, \mathbf{R})$). It is clear that $\widehat{\Gamma}((-Id, -1), x) = x$ for all $x \in S^7$, so $\widehat{\Gamma}$ actually descends to an action of

$$\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{Sp}(1)/(\mathbf{Z}/2\mathbf{Z})\,,$$

on S^7. We have

THEOREM 4.4. *This* SO(4) *action on* S^7 *satisfies:*

(1) *It is a hyperpolar isometric action of cohomogeneity 1, with space of orbits the interval* $[0, \frac{\pi}{2}]$.

(2) *The two exceptional orbits are both diffeomorphic to* $P^2_{\mathbf{R}} \times S^3$ *and both are minimally embedded in* S^7.

(3) *The principal orbits are diffeomorphic to* $F^3(2,1) \times S^3$.

(4) *The square of the distance functions to the exceptional orbits are both Bott-Morse functions.*

(5) *The union of the two exceptional orbits, both copies of* $P^2_{\mathbf{R}} \times S^3$, *is the Spanier-Whitehead dual of one principal orbit* $F^3(2,1) \times S^3$.

We notice that the action of $SO(n+1)$ on $\mathbf{C}^{n+1}$ considered in Section 2 also provides, when $n = 3$, an isometric action of cohomogeneity 1 of SO(4) on S^7. However, in this case the two special orbits are the inverse images of the quadric Q and the real projective space $\Pi \cong P^3_{\mathbf{R}}$ under the projection $S^7 \to P^3_{\mathbf{C}}$. So this action is not equivalent to the "twistorial" one given by Theorem 4.4.

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