2. ON THE GEOMETRY OF P_C^n

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is totally geodesic in $P^n_{\mathbf{C}}$, since it is a complex projective line. Therefore $\mathcal{L}$ intersects Π transversally in a real projective line. This implies that the normal map $\mathcal{N}_Q$ is a diffeomorphism from the open disk bundle in $\nu(Q)$ of radius $\frac{\pi}{2}$ into $P^n_{\mathbf{C}} \setminus \Pi$. The union of all closed geodesic segments normal to Q of length $\frac{\pi}{2}$ fill up all of $P^n_{\mathbf{C}}$. Thus the distance from a point $p \in P^n_{\mathbf{C}} \setminus (\Pi \cup Q)$ to Q is exactly the length of the unique geodesic segment joining p and the unique point $q \in Q$ such that this segment is orthogonal to Q. Hence every tubular neighbourhood of Q in $P^n_{\mathbf{C}}$, of diameter less than $\frac{\pi}{2}$, is diffeomorphic to $P^n_{\mathbf{C}} \setminus \Pi$. $\square$

We remark that one has a construction for the Milnor fibre F of the Fermat polynomial $\mathbf{F}^n_2$ in the spirit of Theorem (1.5), since F can be regarded as the open mapping cylinder of the fibration

$$V_{n+1,2} \cong \mathrm{SO}(n+1,\mathbf{R})/\mathrm{SO}(n-1,\mathbf{R}) \longrightarrow \mathrm{SO}(n+1,\mathbf{R})/\mathrm{SO}(n,\mathbf{R}) \cong S^n,$$

where $V_{n+1,2}$ is the aforementioned Stiefel manifold.

2. On the geometry of $P^n_{\mathbf{C}}$

We now look more carefully at the decomposition of $P^n_{\mathbf{C}}$ arising from the double fibration (1.4). For this, it is convenient to look at two other interesting foliations that arise naturally from the double fibration (1.4), and from other considerations too.

The first foliation $\mathcal{F}_1$ is actually defined on $P^n_{\mathbf{C}} \setminus \Pi$ and its leaves are the fibres of π_1, which are 2-disks transversal to Q, by Theorem 1.5. By construction, each leaf of $\mathcal{F}_1$ is transversal to all the manifolds $F^{n+1}_+(2,1) \times t \subset P^n_{\mathbf{C}}$ for $t \in (0,1)$, intersecting each in a copy of $P^1_{\mathbf{R}}$ and approaching Π as $t \to 1$. Let us construct this foliation in a different way. We endow $P^n_{\mathbf{C}}$ with the Fubini-Study metric. From the proof of Theorem 1.5 we know that the normal map $\mathcal{N}_Q$ of Q induces a diffeomorphism between the open disk bundle of radius $\pi/2$ and $P^n_{\mathbf{C}} \setminus \Pi$. The leaves of $\mathcal{F}_1$ are the images of the normal disks. Since the conjugation $j\colon P^n_{\mathbf{C}} \to P^n_{\mathbf{C}}$ is an isometry, we have that a projective line $\mathcal{L}$ in $P^n_{\mathbf{C}}$ intersects Q at two conjugate points iff it is orthogonal to Q, and this happens iff $\mathcal{L}$ can be defined by equations with real coefficients. So we call these **CR**-lines. If two distinct **CR**-lines intersect, they do so in a point in $\Pi \cong P^n_{\mathbf{R}}$. Also, each **CR**-line $\mathcal{L}$ meets Π in a real projective line, which is an equator of $\mathcal{L}$. Since all complex lines in $P^n_{\mathbf{C}}$ are totally geodesic, the real projective line $\mathcal{L} \cap \Pi$ is a geodesic in $P^n_{\mathbf{C}}$, at equal distance $\pi/2$ from both intersection points in $\mathcal{L} \cap Q$. This divides $\mathcal{L}$ into two round disks of maximal diameter, orthogonal to Q. One can prove that through each point in $P^n_{\mathbf{C}} \setminus \Pi$ there passes a unique **CR**-line, hence these lines foliate this space. Therefore the open disks into which the **CR**-lines split fill out the whole of $P^n_{\mathbf{C}} \setminus \Pi$, they are totally geodesic in $P^n_{\mathbf{C}}$ and orthogonal to Q, thus providing a fibre bundle decomposition of $P^n_{\mathbf{C}} \setminus \Pi$, equivalent to the open disk bundle of the normal bundle $\nu(Q)$ of Q in $P^n_{\mathbf{C}}$. By construction, the closure of each leaf in $P^n_{\mathbf{C}}$ is obtained by attaching to the leaf a real projective line $P^1_{\mathbf{R}} \subset \Pi$, which is its boundary (or limit set). This circle (a real projective line in Π) is invariant by conjugation and it is an equator of a unique **CR**-line, therefore it is also a closed geodesic for the Fubini-Study metric of $P^n_{\mathbf{C}}$.

In the case of the foliation $\mathcal{F}_2$, the leaves are the fibres of π_2, up to isotopy. They are transverse to $F^{n+1}_{+}(2,1) \times t$, for every $t \in (0,1)$, and these leaves are also transverse to Π. We can describe this foliation more precisely as follows. Given $z \in \Pi$, we let $\mathcal{P}_z$ be the pencil of real projective lines in Π passing through z. Note that the tangent vectors at z to the lines of this pencil give the tangent space of Π at z. Let l_z be one of the lines of the pencil $\mathcal{P}_z$. Its complexification is a projective line L_z in $P^n_{\mathbf{C}}$ defined by an equation with real coefficients, invariant under conjugation. This implies that L_z intersects Q at two points w_1 and w_2, which are conjugate; the intersection $L_z \cap Q$ is necessarily orthogonal and l_z is an equator in L_z. Thus, there is a segment $\hat{l}_z$, half of a real projective line (a circle) in L_z, joining the points w_1, z and w_2. This line is orthogonal to Π and to Q, it is geodesic in $P^n_{\mathbf{C}}$ and has length π, by the minimality of L_z. Doing this for all lines in the pencil $\mathcal{P}_z$, we get an open n-disk of radius $\pi/2$ in $P^n_{\mathbf{C}}$, orthogonal to Π at z, filled by geodesics in $P^n_{\mathbf{C}}$ of length $\pi/2$ and intersecting Q orthogonally. Thus the normal map $\mathcal{N}_\Pi$ is regular for vectors of norm $< \pi/2$. The leaves of $\mathcal{F}_2$ are the images under $\mathcal{N}_\Pi$ of the fibres of the open disk normal bundle of $\Pi \subset P^n_{\mathbf{C}}$ of radius $\frac{\pi}{2}$.

There is another interesting way of thinking about this foliation, up to isotopy, which helps to understand the way in which its leaves approach Q. By Corollary 1.2 we have that $P^n_{\mathbf{C}} \setminus Q$ is the Milnor fibre $F := \{z_0^2 + \cdots + z_n^2 = 1\}$ divided by the monodromy $(z_1, \ldots, z_n) \mapsto (-z_1, \ldots, -z_n)$. The fibre F is the tangent bundle of the n-sphere, so it has a natural foliation by leaves diffeomorphic to n-planes. These planes can be described as follows. Let us decompose each $Z := (z_1, \ldots, z_n)$ into its real and imaginary parts, $Z = U + iV$. The fibre F is the set $(U, V) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ such that $||U|| \geq 1$, $||U||^2 - ||V||^2 = 1$ and $U \perp V$. If $||U|| = 1$, then we are on the n-sphere and $||V|| = 0$. Given a fixed $U_0 \in S^n \subset \mathbf{R}^{n+1}$, its "tangent space" is the plane defined as follows: for each $\lambda \in \mathbf{R}$ with $\lambda > 1$, let $S_\lambda(U_0)$ be the

$(n-1)$-sphere in the affine n-plane perpendicular to λU_0, consisting of all vectors V such that the vector $Z = \lambda U_0 + iV$ is in F; these must satisfy $\|V\|^2 = \lambda^2 - 1$. The radius of the sphere $S_\lambda(U_0)$ grows with λ, while for $\lambda = 1$ the corresponding "sphere" is just one point. For a given $U_0 \in S^n$, let us denote by $\mathcal{L}(U_0)$ the union of all these $(n-1)$-spheres $S_\lambda(U_0)$, for all $\lambda \geq 1$. Then $\mathcal{L}(U_0)$ is a copy of $\mathbf{R}^n$ embedded in F as a component of the 2-sheeted hyperboloid consisting of $\mathcal{L}(U_0) \cup \mathcal{L}(-U_0)$. The monodromy map interchanges these two sheets of the hyperboloid, so their image in $P^n_{\mathbf{C}}$ is a manifold diffeomorphic to a plane, that we denote by $\mathcal{F}(U_0)$. By the uniqueness of the tubular neighbourhood, these are the leaves of $\mathcal{F}_2$ up to isotopy.

From this description of $\mathcal{F}_2$ one can see the way in which the leaves approach Q. In fact, let us denote by $S'_\lambda(U_0)$ the image of the sphere $S_\lambda(U_0)$ in $P^n_{\mathbf{C}}$. It lies in $\mathcal{F}(U_0)$. Let $\gamma_\lambda(U_0)$ be the intersection of the unit sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ with the real half cone over $S_\lambda(U_0)$ with vertex at 0. The image of $\gamma_\lambda(U_0)$ in $P^n_{\mathbf{C}}$ is also $S'_\lambda(U_0)$. The sphere $\gamma_\lambda(U_0)$ is the set of vectors $(\frac{\lambda}{\sqrt{2\lambda^2-1}} U_0, \frac{1}{\sqrt{2\lambda^2-1}} V)$ with $(\lambda U_0, V)$ in $S_\lambda(U_0)$. Therefore the limit of $\gamma_\lambda(U_0)$ is the set of vectors $(\frac{1}{\sqrt{2}} U_0, \frac{1}{\sqrt{2}} v)$ where v is $V/\|V\|$, with V as above. Since the vectors $\frac{1}{\sqrt{2}} U_0$ and $\frac{1}{\sqrt{2}} v$ have equal length, the image $\Lambda(U_0)$ in $P^n_{\mathbf{C}}$ of this limit set is in Q, and it is a $(n-1)$-sphere. By continuity, the limit set of $S'_\lambda(U_0)$ in $P^n_{\mathbf{C}}$ is also $\Lambda(U_0)$. Since the conjugate of the vector (U, V) is $(U, -V)$, the sets $\gamma_\lambda(U_0)$ and their limit, are invariant under conjugation. Hence $\Lambda(U_0)$ is also invariant by conjugation.

Let us summarize the previous discussion in the following

PROPOSITION 2.1. *The double fibration (1.4) induces two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ such that:*

i) *The first one $\mathcal{F}_1$ is defined on $P^n_{\mathbf{C}} \setminus \Pi$; its leaves are embedded copies of $\mathbf{R}^2$, orthogonal to Q, which are the images under the normal map of Q of the fibres of the normal disk bundle of Q of radius less than $\frac{\pi}{2}$. The closure of each such leaf is a closed 2-disk that meets Π orthogonally in a projective line which is a closed geodesic in $P^n_{\mathbf{C}}$. For each pair of conjugate points in Q, the corresponding leaves are naturally glued together along their common limit set in Π, forming a complex projective line defined by real coefficients.*

ii) *The second foliation $\mathcal{F}_2$ is defined on $P^n_{\mathbf{C}} \setminus Q$; its leaves are embedded n-disks, orthogonal to Π, which are the images under the normal map of Π of the fibres of the normal disk bundle of Π of radius less than $\frac{\pi}{2}$. The closure of each such leaf is a closed n-disk that meets Q orthogonally in a $(n-1)$-sphere, invariant under complex conjugation.*

We notice that the previous discussion also proves the following fact, that we state as a proposition. We recall that given a Riemannian submanifold N of $P^n_{\mathbf{C}}$, its *focal points* are the critical values of the normal map of N, see [Mi1].

PROPOSITION 2.2. *The real projective space $\Pi \cong P^n_{\mathbf{R}}$, consisting of the points in $P^n_{\mathbf{C}}$ with homogeneous real coordinates, is the set of focal points of the quadric Q defined by the Fermat polynomial $z_0^2 + \cdots + z_n^2 = 0$. Conversely, the quadric Q is the set of focal points of Π.*

Thus, both manifolds Q and Π can be regarded as *caustics* in $P^n_{\mathbf{C}}$, i.e. they are the critical values of the Lagrangian maps defined by the corresponding co-normal maps of Π and Q, respectively (see [AG]).

Let us consider now the action of $\mathrm{SO}(n+1, \mathbf{R})$ on $P^n_{\mathbf{C}}$, regarded as a subgroup of the complex orthogonal group $O(n+1, \mathbf{C})$. This action leaves Q invariant and it is by isometries with respect to the Fubini-Study metric. An isometry of $P^n_{\mathbf{C}}$ that leaves Q invariant necessarily carries the set of focal points of Q into itself. Hence Π is also an invariant set for the action of $\mathrm{SO}(n+1, \mathbf{R})$. We know already that Q is the Grassmannian $G_{n+1,2} \cong \mathrm{SO}(n+1, \mathbf{R})/(\mathrm{SO}(n-1, \mathbf{R}) \times \mathrm{SO}(2, \mathbf{R}))$, so the action of $\mathrm{SO}(n+1, \mathbf{R})$ is transitive on Q. Thus Q is one single orbit, and so is Π. Let us look at the orbit of a point $w \in P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$. We claim that its orbit is the manifold $(F_+^{n+1}(2,1) \times t)$ passing through w. For this we use again the normal map

$$\mathcal{N}_Q \colon \nu(Q) \to P^n_{\mathbf{C}} .$$

By the previous discussion, this map is a diffeomorphism from the open disk bundle in $\nu(Q)$ of radius $\frac{\pi}{2}$ into $P^n_{\mathbf{C}} \setminus \Pi$ and the images of the fibres are the leaves of $\mathcal{F}_1$. Hence each point $w \in P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$ is in the image of the normal map $\mathcal{N}_Q$, i.e., there is a (unique) vector $v_w \in \nu(Q)$ normal to Q, such that $w = \mathcal{N}_Q(v_w)$; the norm of v_w equals the distance $d_w = d(w, Q)$ from w to Q, which is > 0 and $< \pi/2$. That is, w corresponds, via $\mathcal{N}_Q$, to a point in the sphere bundle $S_{d_w}(\nu(Q))$ of radius d_w in $\nu(Q)$. We claim that the $\mathrm{SO}(n+1, \mathbf{R})$-orbit $\mathcal{O}_w$ of w is the image of this sphere bundle, i.e. $\mathcal{O}_w = \mathcal{N}_Q(S_{d_w}(\nu(Q)))$. For this we notice that the group $\mathrm{SO}(n+1, \mathbf{R})$ also acts on the tangent bundle $TP^n_{\mathbf{C}}$ via the differential, and this action preserves the (C^∞) splitting $TP^n_{\mathbf{C}}|_Q \cong TQ \oplus \nu(Q)$. This induces an action of $\mathrm{SO}(n+1, \mathbf{R})$ on the normal bundle $\nu(Q)$ of Q, and this action is isometric and commutes with $\mathcal{N}_Q$, proving the claim. Hence the $\mathrm{SO}(n+1, \mathbf{R})$-orbits are all manifolds $(F_+^{n+1}(2,1) \times t)$, for some $t \in (0,1)$, with two exceptional orbits which are

Q and Π, corresponding to $t=0$ and $t=1$. By [HL; 1.1], this implies that Q and Π are minimal submanifolds of $P^n_{\mathbf{C}}$, which is obvious for Q, being a complex submanifold. The orbits of maximal dimension, which in this case are diffeomorphic to $F^{n+1}_{+}(2,1)$, are called *principal orbits*.

The previous arguments also show that each $\mathrm{SO}(n+1,\mathbf{R})$-orbit in $P^n_{\mathbf{C}}$ is at constant distance from Q, and also from Π, and these distances go from 0 to $\frac{\pi}{2}$. This proves that the space of $\mathrm{SO}(n+1,\mathbf{R})$-orbits in $P^n_{\mathbf{C}}$ is the interval $[0,\frac{\pi}{2}]$, with the two special orbits corresponding to the endpoints of the interval. But one can actually be more precise about this statement. Let us consider again the geodesic $\hat{l}_z$ described above, in the construction of the foliation $\mathcal{F}_2$. In fact we are interested in half of this geodesic segment. To construct this "half geodesic segment", that we shall denote by $\check{l}$, we can start with any complex projective **CR**-line $\mathcal{L}$. This line intersects Π in a real projective line, and it meets Q orthogonally at two conjugate points, say w and $\overline{w}$. Now we choose a point $z_0 \in \Pi \cap \mathcal{L}$. Then $\check{l}$ is the geodesic (of length $\frac{\pi}{2}$) in $\mathcal{L}$ joining the points z_0 and w, and it is a geodesic in $P^n_{\mathbf{C}}$ because $\mathcal{L}$ is totally geodesic. This geodesic $\check{l}$ starts at $z_0 \in \Pi$ and ends at $w \in Q$. Hence it meets each $\mathrm{SO}(n+1,\mathbf{R})$-orbit orthogonally in exactly one point, since the orbits are the level sets of the function distance to Π. Hence $\check{l}$ parametrizes the orbits of $\mathrm{SO}(n+1,\mathbf{R})$. This shows that the $\mathrm{SO}(n+1,\mathbf{R})$-action on $P^n_{\mathbf{C}}$ is a hyperpolar isometric action of cohomogeneity 1, which is already well known (see for instance [HPTT, Ko]). In fact, *cohomogeneity 1* means that the principal orbits have codimension 1, and we know that this happens in our case. An isometric action is said to be *polar* if there exists a closed, connected submanifold Σ that meets all orbits orthogonally. In our case this can be, for instance, the complete geodesic in $\mathcal{L}$ determined by $\check{l}$. Such a manifold is called a *section*. If one can choose such a section to be also flat, one says that the action is *hyperpolar*. This is obviously satisfied in our case since the section is a geodesic.

We have thus proved the following

THEOREM 2.3.

i) *The natural* $\mathrm{SO}(n+1,\mathbf{R})$-*action on* $P^n_{\mathbf{C}}$ *is an isometric, hyperpolar action of cohomogeneity* 1, *whose space of orbits is the interval* $[0,\pi/2]$. *A section for this action (i.e. a submanifold that intersects transversally each orbit at exactly one point) can be constructed by considering some (any)* **CR**-*line* $\mathcal{L}$, *choosing a point* $z \in \mathcal{L} \cap \Pi$ *and taking the geodesic (a circle) in* $\mathcal{L}$ *that passes through* z *and the two points where* $\mathcal{L}$ *meets* Q.

ii) *There are three orbit types: two special orbits, Q and Π, which correspond to the endpoints $\{0, \pi/2\}$, and the principal orbits, which are copies of the partial flag manifold,*

$$F_+^{n+1}(2,1) \cong \mathrm{SO}(n+1, \mathbf{R})/(\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}),$$

of oriented 2-planes in $\mathbf{R}^{n+1}$ and lines in these planes. The manifold $F_+^{n+1}(2,1)$ is diffeomorphic to the unit sphere normal bundle of Q in $P^n_{\mathbf{C}}$, and also to the unit sphere tangent bundle of $P^n_{\mathbf{R}}$. Each of the two special orbits is the set of focal points of the other, and they are minimal submanifolds of $P^n_{\mathbf{C}}$.

iii) *The complex projective lines in $P^n_{\mathbf{C}}$ whose homogeneous coordinates are real, i.e. the* **CR***-lines, foliate $P^n_{\mathbf{C}} \setminus \Pi$ and they are everywhere transversal to the orbits of* $\mathrm{SO}(n+1, \mathbf{R})$ *(away from Π). In particular, they are orthogonal to Q.*

iv) *The real projective space $\Pi \cong P^n_{\mathbf{R}}$ is embedded in $P^n_{\mathbf{C}}$ so that its normal bundle is isomorphic to its tangent bundle. Its "tangent spaces" naturally define a foliation of $P^n_{\mathbf{C}} \setminus Q$ by embedded copies of $\mathbf{R}^n$, which are everywhere transversal to the orbits of* $\mathrm{SO}(n+1, \mathbf{R})$ *(away from Q). In particular, they are orthogonal to Π.*

We now let $q \colon P^n_{\mathbf{C}} \to [0, \pi/2] \subset \mathbf{R}$ be the function $q(Z) = [d(Z, Q)]^2$, i.e. q is the square of the distance to Q. It is clear that q is constant along the $\mathrm{SO}(n+1, \mathbf{R})$-orbits, which are its level sets. Hence q has the two special orbits Q and Π as critical set. It is clear that if Σ is a small disk in $P^n_{\mathbf{C}}$ orthogonal to Q (or to Π), then the restriction of q to Σ is the ordinary quadratic map, so it is a Morse function on Σ. This means, by definition, that q is a Bott-Morse function. We have thus obtained the following results, motivated by [DR]:

COROLLARY 2.4. *The map q is a Bott-Morse function, whose level surfaces are the orbits of* $\mathrm{SO}(n+1, \mathbf{R})$ *and the critical set consists of the two special orbits Q and $\Pi \cong P^n_{\mathbf{R}}$.*

Of course one can replace the function q above by $p(Z) = [d(Z, \Pi)]^2$, which is also a Bott-Morse function.

COROLLARY 2.5. *Let* $\pi\colon S^{2n+1} \to P^n_{\mathbf{C}}$ *be the Hopf fibration. Then the composition* $p \circ \pi\colon S^{2n+1} \to [0, \pi/2] \subset \mathbf{R}$ *is a Bott-Morse function. The critical set has two components, which are* S^1*-bundles over the two special orbits in Theorem 2.2. One of these is the Stiefel manifold* $V_{n+1,2} \subset S^{2n+1}$ *of real oriented 2-planes in* $\mathbf{R}^{n+1}$*, diffeomorphic to the link of the affine Fermat quadric, the other is the unique non-trivial* S^1*-bundle over* $P^n_{\mathbf{R}}$*. (Both of these are minimally embedded submanifolds of* S^{2n+1}*, by* [HL].*)*

REMARKS 2.6.

i) We notice that if we let $S^2(P^1_{\mathbf{C}})$ be the symmetric product $(P^1_{\mathbf{C}} \times P^1_{\mathbf{C}})/I$, where I is the involution $I(x,y) = (y,x)$, then there is a canonical holomorphic surjection $p\colon P^1_{\mathbf{C}} \times P^1_{\mathbf{C}} \to S^2(P^1_{\mathbf{C}})$ taking (x,y) to the point $[(x,y)]$ in $S^2(P^1_{\mathbf{C}})$. This induces an isomorphism $S^2(P^1_{\mathbf{C}}) \cong P^2_{\mathbf{C}}$. Hence, every identification $Q \cong P^1_{\mathbf{C}}$ also determines an analytic isomorphism $S^2(Q) \cong P^2_{\mathbf{C}}$, where the conic Q in $P^2_{\mathbf{C}}$ is the image of the diagonal Δ. This is, essentially, a special case of the projective Vieta Theorem, which says that $P^n_{\mathbf{C}}$ is the n^{th} symmetric power of $P^1_{\mathbf{C}}$. A real version of this result was proved by Arnold in [Ar3; Th. 2].

ii) Let us denote by j the antipodal map in $P^1_{\mathbf{C}} \cong \mathbf{C} \cup \{\infty\}$. This is given by $j(z) = -1/\bar{z}$, and is a fixed point free involution of $P^1_{\mathbf{C}}$. The anti-diagonal (the graph of the antipodal map) in $P^1_{\mathbf{C}} \times P^1_{\mathbf{C}}$ is given by

$$\Delta^{-1} := \{(z, -1/\bar{z})\} \, .$$

This gives a copy of $P^1_{\mathbf{C}}$ anti-holomorphically embedded in $(P^1_{\mathbf{C}} \times P^1_{\mathbf{C}}) \setminus \Delta$. It is clear that Δ^{-1} is invariant under the involution $I(x,y) = (y,x)$ of $(P^1_{\mathbf{C}} \times P^1_{\mathbf{C}})$. Thus $\Delta^{-1} := \{(z, -1/\bar{z})\}$ is projected onto a smooth copy of $P^2_{\mathbf{R}}$ in $P^2_{\mathbf{C}}$, disjoint from Q. Hence the identification ϕ of $P^1_{\mathbf{C}}$ with $Q \subset P^2_{\mathbf{C}}$ also determines, canonically, a copy of the real projective space $P^2_{\mathbf{R}}$ in $P^2_{\mathbf{C}} \setminus Q$, together with an involution of $P^2_{\mathbf{C}}$ whose fixed point set is this $P^2_{\mathbf{R}}$. If Q is the Fermat conic, $\{z_1^2 + z_2^2 + z_3^2 = 0\}$, then this embedding of $P^2_{\mathbf{R}}$ in $P^2_{\mathbf{C}}$ is the usual one.

iii) We notice that, also for $n = 2$, every diffeomorphism $f\colon Q \to Q$ extends canonically to a diffeomorphism $\widetilde{f}\colon P^2_{\mathbf{C}} \to P^2_{\mathbf{C}}$, and this extension is functorial, i.e., $\widetilde{f_2 \circ f_1} = \widetilde{f}_2 \circ \widetilde{f}_1$ (cf. [Gh]). In fact, through every point $u \in P^2_{\mathbf{C}} - Q$, there are two tangents to Q, which determine points $\phi(a_1), \phi(a_2)$ in Q. Then $\widetilde{f}(u)$ is the point of intersection of the lines tangent to Q at the points $f(\phi(a_1))$ and $f(\phi(a_2))$. A consequence of these remarks is that if G is a group acting on Q, then the G-action extends to $P^2_{\mathbf{C}}$. In particular, if G is $\mathrm{SO}(3, \mathbf{R})$, acting on $Q \cong S^2$ by rotations, its extension to $P^2_{\mathbf{C}}$ is the action that we considered in Sections 2 and 3 above. Similarly, if G is $\mathbf{Z}/2\mathbf{Z}$ acting on Q as the antipodal map, then the corresponding extension to $P^2_{\mathbf{C}}$ is given by complex conjugation.

3. $P^2_{\mathbf{C}}$ AND THE 4-SPHERE S^4

The previous discussion, restricted to $n = 2$ and compared to the cohomogeneity 1 isometric action of $\mathrm{SO}(3,\mathbf{R})$ on S^4 constructed in [HL], motivates an equivariant version of the Arnold-Kuiper-Massey theorem [Ar1, Ar2, Ku, Ma1], saying that $P^2_{\mathbf{C}}$ modulo conjugation is the 4-sphere. In this section we give a new proof of this theorem. We construct an explicit algebraic map $\Phi \colon P^2_{\mathbf{C}} \to S^4$, which is equivariant with respect to the cohomogeneity 1 isometric actions of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$ and S^4 and induces a diffeomorphism $P^2_{\mathbf{C}}/\textit{conjugation} \cong S^4$.

We start by recalling the $\mathrm{SO}(3,\mathbf{R})$-action on S^4, as explained by Hsiang and Lawson in [HL; Example 1.4].

Let $\mathcal{S}$ be the vector space of real 3×3, traceless and symmetric matrices. As a real vector space $\mathcal{S}$ is $\mathbf{R}^5$, and it can be equipped with a metric given by the inner product $(A,B) \mapsto \mathrm{trace}(AB)$. Let $\mathcal{S}^{(4)}$ be the space of matrices in $\mathcal{S}$ with norm 1. One has an obvious diffeomorphism $S^4 \cong \mathcal{S}^{(4)}$, which becomes isometric if we endow S^4 with its usual round metric and $\mathcal{S}^{(4)}$ with the metric given by the inner product in $\mathcal{S}$. We shall identify these two spaces in the sequel, denoting both of them by S^4 indistinctly. The group $\mathrm{SO}(3,\mathbf{R})$ acts on $\mathcal{S}$ by $A \mapsto O^tAO$, where O^t is the transposed matrix (which is equal, in our case, to O^{-1}). This induces an isometric action Γ of $\mathrm{SO}(3,\mathbf{R})$ on S^4. This action on S^4 has two disjoint copies of $P^2_{\mathbf{R}}$ as special fibres (see the remark at the end of this section). The space of orbits is the interval $[0,1]$, with the endpoints giving the special orbits. Each principal orbit (i.e. the orbits of highest dimension) is a flag manifold

$$F^3(2,1) \cong \mathrm{SO}(3,\mathbf{R}) / (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \cong L(4,1) / (\mathbf{Z}/2\mathbf{Z}),$$

of pairs (P,l) with P a plane in $\mathbf{R}^3$ and l line in P, where $L(4,1)$ is the lens space $S^3 / (\mathbf{Z}/4\mathbf{Z}) \cong \mathrm{SO}(3,\mathbf{R}) / (\mathbf{Z}/2\mathbf{Z})$.

Let us give a similar description of $P^2_{\mathbf{C}}$. Let

$$\mathfrak{H}(3,\mathbf{C}) = \{H \in M(3,\mathbf{C}) \mid H = H^*\}$$

be the space of complex 3×3 Hermitian matrices, where $H^* = \overline{H}^t$ is the adjoint matrix of H, obtained by first conjugating each entry of H and then