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§4. REAL ALGEBRAIC MAPS

We are going to show that the entropy of a real algebraic map of "algebraic degree p" cannot exceed $n \log p$, where n is the dimension. One way of approach is to complexify the whole situation, i.e. to take the Zariski closure of the graph of the map, and to apply reasoning from the previous section. This enables us to solve both problems: to introduce the notion of degree and to prove the inequality $h \leq n \log p$.

In order to avoid passing to complex numbers and to make the proof applicable to piecewise algebraic (say, piecewise linear) maps we shall now present a different argument based on Bézout's theorem.

Let us start for the sake of simplicity with a map f given by two polynomials in $\mathbf{R}^2$ of degree p. Suppose that f sends a square $S \subset \mathbf{R}^2$ into itself and try to estimate the entropy of $f\colon S \to S$. Divide S into pieces S_j of size $\leq \epsilon$ by straight lines l_i, $i = 1, 2, \ldots, r$; see Figure 1.

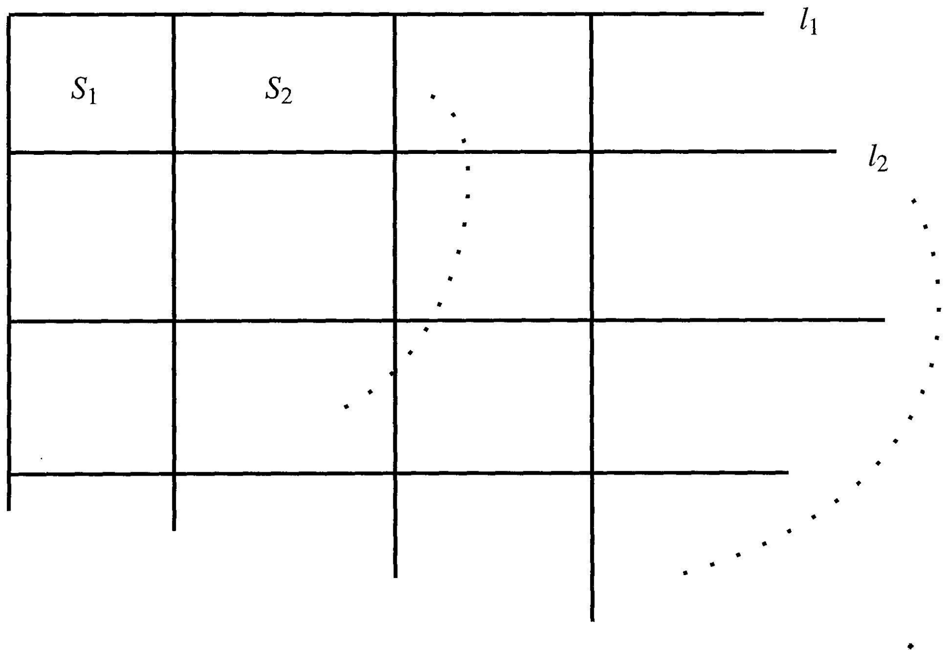

FIGURE 1

Try now to cover the iterated graph $(\Gamma_f)_k$ by products of these pieces. Observe that a product $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_k} \subset S^k$ intersects $(\Gamma_f)_k$ if and only if the intersection $S_{j_1} \cap f^{-1}(S_{j_2}) \cap \cdots \cap f^{-(k-1)}(S_{j_k})$ is not empty. Now let us estimate the number N_k of all such non-empty intersections. This number is not greater than the number of components in

$$S \setminus \bigcup_{\substack{1 \leq \mu \leq k \\ 1 \leq i \leq r}} f^{-\mu}(l_i)$$

and thus N_k cannot exceed 4 times the number of all pairwise intersections of all lines $f^{-\mu}(l_i)$, provided the map $f\colon S \to S$ was injective. The system of lines $\{l_i\}$ can be represented as the zero-set of a polynomial of degree r and thus the system $\{f^{-\mu}(l_i)\}$ given by a polynomial of degree $k \cdot r \cdot p^k$; by Bézout's theorem the number of intersections is not greater than $k^2r^2p^{2k}$. So $N_k \leq 4k^2r^2p^{2k}$ and $\lim_{k\to\infty} \frac{1}{k} \log N_k \leq 2\log p$.

In the general case (when $n > 2$, or $n = 2$ and f is not injective), there appears a complication pointed out by J. Milnor (and communicated to me by Newhouse): the components in the complement $S \setminus \bigcup f^{-\mu}(l_i)$ can contain no points of intersections of lines (or surfaces when $n > 2$). A typical 'bad' picture for $n = 3$ is shown in Figure 2.

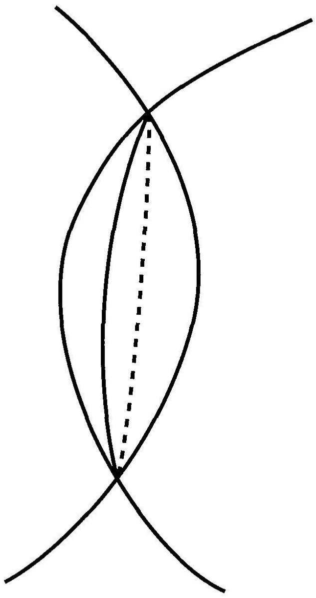

FIGURE 2

But in any case, each component must contain in its boundary a component of an algebraic variety determined by certain intersections of $f^{-\mu}(l_i)$; to estimate the number of those, Milnor suggested using his (and Thom's) theorem (see [3]):

The number of all components of the zero-set of a system of polynomials of degree D in $\mathbf{R}^n$ is not greater than $(2D)^n$.

(The actual Milnor inequality is more precise and also takes into account the Betti numbers in positive dimensions.)

It seems more natural to apply the Milnor theorem not in the space X itself, but in the product X^k, in particular when we deal with an algebraic graph. Unfortunately, the direct application leads to the weaker estimate $h \leq n \log(2p)$, due to the coefficient 2 in the Milnor theorem. But in our rather special situation, this 2 can easily be removed and we always have $h \leq n \log p$.

Periodic points

The argument above is the same as in the Artin-Mazur theorem on periodic points [1]: the number of *isolated* periodic points of a map of degree p cannot exceed $(np)^k$, where k is the period. The number n is the dimension of the Euclidean (or projective) space in which the manifold X is realized, and p is the degree of polynomials defining the graph $\Gamma \subset X \times X \subset \mathbf{R}^{2n}$ of the map. The points of period k correspond to the intersection of $\Gamma_k \subset \mathbf{R}^{kn}$ with the preimage of the diagonal in $X \times X$ by the projection of X^k to the product of its first and last factors. Artin and Mazur make use of Bézout's theorem, which immediately yields the needed estimate. Notice that the Milnor-Thom inequality implies an analogous estimate for all Betti numbers of the sets of periodic points.

Artin and Mazur combine their estimate with the Nash theorem on approximation of a smooth map by algebraic ones (i.e. with algebraic graphs) and conclude: For a dense set of smooth maps, the number of *isolated* points k is not greater than $(\text{const})^k$. Omitting 'isolated' seems not completely trivial (though geometrically obvious) in the pure algebraic situation. (One even expects a 'generic algebraic' map to have no invariant algebraic manifolds of positive dimension, unlike the smooth case where invariant manifolds can be persistent.)

The following argument, communicated to me by Newhouse, allows one to get rid of the 'isolated'.

A map $X \to X$, $X \subset \mathbf{R}^n$, can be extended to a neighbourhood of X by a map F strongly expanding in directions normal to X. The invariant manifold X of this extension is stable under small perturbation of F and thus the general situation is reduced to the simple case of polynomial maps in $\mathbf{R}^n$.

Geometric approach

The last remark undermines the role of algebraic maps in differential dynamics (but, of course, algebraic dynamics is in many respects more interesting than differential anyway) and we can go even further replacing degree by a kind of geometric complexity (in spirit of Thom) of a smooth map, making use of a quantitative form of the Thom transversality theorem instead of Bézout's theorem. The quantitative transversality can be used also for counting periodic orbits of a vector field and periodic points of a transformation preserving an additional structure (volume, symplectic form, etc.) and it yields the Artin-Mazur estimate for dense sets of such maps. Unfortunately, the detailed proof (at least the one I know of) is more lengthy than the algebraic one, and I shall treat the subject somewhere else.

REMARK. The quantitative transversality theory has been developed by Y. Yomdin (see p. 124 in [5′] for a brief introduction) but does not suffice, as it stands, for the diff-version of the Artin-Mazur theorem. In fact, one needs here an adequate notion of genericity (compare remark on p. 31 in [5′]) as is shown in [1′] for smooth maps. I have never returned to this issue and can only conjecture the extension (and sharpening) of the corresponding results in [1′] to structure preserving maps and/or vector fields.

§5. QUASICONFORMAL MAPS

For a smooth map $f\colon X \to Y$ from one oriented n-dimensional Riemannian manifold into another, we denote by $D_x f$ its differential at x, by $\|D_x f\|$ the norm of the differential, by $\det D_x f$ its Jacobian, and by $\lambda_x f$ the ratio $\|D_x f\|^n / \det D_x f$ called the *conformal dilation* at $x \in X$. A map is called λ-quasiconformal if, for almost all x, the differential $D_x f$ exists, $\det D_x f$ does not vanish, and $\lambda_x f \leq \lambda$. A quasiconformal map must have locally positive degree. If $n = 1$, each locally diffeomorphic map is conformal (i.e. 1-quasiconformal).

When $n = 2$, conformal maps are complex analytic and for $n > 2$ all conformal maps are locally diffeomorphic. In particular, when $n > 2$, every non-injective conformal endomorphism is conjugate to a homothety of a flat Riemannian manifold. When $\lambda > 1$, there are (not locally injective) λ-quasiconformal maps in all dimensions $n > 1$. They are locally homeomorphic outside a codimension 2 branching set. At that set, they are never $(n > 2)$ smooth.