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§6. Mean curvature

Let X be a closed n-dimensional manifold with a Riemannian metric g. Suppose that iterated graphs $\Gamma_k \subset X^k$ are smooth of dimension n. Denote by $\mathrm{Cu}(\gamma)$, $\gamma \in \Gamma_k$, the absolute value of the mean curvature of Γ_k at γ. Set

$$\mathrm{lome}_g \Gamma = \limsup_{k\to\infty} \frac{1}{k} \log\Big(1 + \int_{\Gamma_k} [\mathrm{Cu}(\gamma)]^n d\gamma\Big) .$$

When Γ_k are minimal and $\mathrm{lome}_g = 0$ we know that $h \le$ "lov".

More generally,

$$h(\gamma) \le \mathrm{lov}\ \Gamma + \mathrm{lome}_g \Gamma . \tag{6.0}$$

Proof. Despite the possible dependence of "lome" upon the choice of g, we can proceed as before and reduce (6.0) to the following local estimate:

Take V in the Euclidean space $\mathbf{R}^{\ell=kn}$ and suppose its boundary does not intersect the ball $B_{2\epsilon}$ centered at $v_0 \in V$. Then

$$\epsilon^{-n} \mathrm{Vol}\ V + \int_V Cu^n(v) dv \ge C_1 \ell^{C_2} , \tag{6.1}$$

where C_1 and C_2 are constants depending only on n.

To prove (6.1) we consider the normal bundle N of V and its canonical map F into $\mathbf{R}^\ell$. The Jacobian J of F at a point $v + \nu t$ (where $v \in V$, and ν is the unit vector at v normal to V) is equal to $\prod_{i=1}^n (1 + k_i t)$, where k_i are the principal curvatures in the direction ν.

If the distance from $v+\nu t$ to V is equal to t, then $1+k_i t \ge 0$, $i = 1, \dots, h$, and so

$$J \le A_n \big(1 + t^n Cu^n(v)\big) . \tag{6.2}$$

Now we observe that every point of the ball B_ϵ can be joined by a shortest normal with V and so

$$C_\ell \epsilon^\ell = \mathrm{Vol}\ B_\epsilon \le A_n C_{\ell-n} \epsilon^{\ell-n} \int_V \big(1 + \epsilon^n Cu^n(v)\big)\, dv ,$$

where C_ℓ and $C_{\ell-n}$ are volumes of unit balls in $\mathbf{R}^\ell$ and $\mathbf{R}^{\ell-n}$. The last inequality implies (6.1) and so (6.0) is proved.

The inequality (6.2) was extended by Karcher and Heinze to general Riemannian manifolds. Discussions with Karcher about such inequalities influenced my reasoning in this section.