3.1 Horizontal and vertical metrics

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Let $f\colon K_\psi \to \mathbf{R}$ be defined by $f(a) = r$ if $a \in X \times \{r\}$. Then f is a continuous surjective map. The preimage of any real number r is $X \times \{r\}$, a topological forest. Furthermore, for any $t \geq 0$, $f \circ \sigma_t = \tau_t \circ f$, where $\tau_t\colon \mathbf{R} \to \mathbf{R}$ is defined by $\tau_t(r) = r + t$.

We extracted above the two properties shared by mapping-telescopes which are really important for our work. We now define a class of spaces which satisfy these two properties, and in particular generalize the mapping-telescopes.

DEFINITION 2.2. Let X be a topological space. Let $(\sigma_t)_{t \in \mathbf{R}^+}$ be a semi-flow on X. Let $f\colon X \to \mathbf{R}$ be a surjective continuous map such that:

1. For any real number r, the *stratum* $f^{-1}(r)$ is a topological forest.
2. For any $t \geq 0$, $f \circ \sigma_t = \tau_t \circ f$, where $\tau_t(r) = r + t$ for any real number r.

Then X is a *forest-stack*, denoted by (X, f, σ_t).

REMARK 2.3. All the strata of a mapping-telescope are homeomorphic. This is not required in the definition of a forest-stack.

As we just saw, a mapping-telescope is an example of a forest-stack. In Section 13, we show that a Cayley complex for the mapping-torus group of an injective free group endomorphism is a mapping-telescope of a forest-map, and thus a forest-stack. The reader can also find there, and in Section 12, an illustration of the horizontal and vertical metrics on forest-stacks, which we are now going to define.

3. METRICS

The aim of this section is to introduce a particular metric on forest-stacks, called the *telescopic metric*. We sometimes deal with metric spaces which are not necessarily connected, for instance forests. In this case, when considering the distance between two points, it will always be tacitly assumed that the two points lie in a same connected component of the space.

3.1 HORIZONTAL AND VERTICAL METRICS

Let us consider a forest-stack $(\widetilde{X}, f, \sigma_t)$, see Definition 2.2. We want to define a natural metric on the orbits of the semi-flow.

DEFINITION 3.1. The *future orbit* $O^+(x)$ of a point x under the semi-flow is the set of points y such that $\sigma_t(x) = y$ for some $t \geq 0$.

The *past orbit* $O^-(x)$ of a point x under the semi-flow is the set of points y such that x is in the future orbit of y.

The *orbit* $O(x)$ of a point x under the semi-flow is the set of points y such that there exists a point z which lies in the future orbit of both x and y.

Let us observe that in general the orbit of a point x strictly contains the union of the future and past orbits of x.

The orbits of the semi-flow are topological trees. This is a straightforward consequence of the semi-conjugacy of the semi-flow with the translations in $\mathbf{R}$ via the map f. Let x, y be any two points in a same orbit of the semi-flow. Assume that x and y lie in a same future orbit of the semi-flow. We consider the orbit-segment between x and y, where an *orbit-segment* is a compact interval contained in the future orbit of some point. The function f is a homeomorphism from this orbit-segment onto an interval of the real line. We define the distance between x and y as the real length of this interval. Assume now that x and y do not lie in a same future orbit. The future orbits of x and y meet at some point z such that the concatenation of the orbit-segment between x and z with the orbit-segment between z and y is an injective path. We then define the distance between x and y as the sum of the distances between x and z and z and y. We have thus defined a distance on the orbits of the semi-flow. This distance is called the *vertical distance*.

DEFINITION 3.2. A *vertical path* in a forest-stack is a path contained in an orbit of the semi-flow. A *vertical geodesic* is an injective vertical path.

A *horizontal path* in a forest-stack is a path contained in a stratum. A *horizontal geodesic* is an injective horizontal path.

DEFINITION 3.3. Let $(\widetilde{X}, f, \sigma_t)$ be a forest-stack. Let $\mathcal{H} = (m_r)_{r\in\mathbf{R}}$ be a collection of metrics on the strata of $\widetilde{X}$. Then $\mathcal{H}$ is a *horizontal metric* if for any $r \in \mathbf{R}$, any $\epsilon > 0$, and any x, y in a same connected component of the stratum $f^{-1}(r)$, there exists $\mu > 0$ such that $0 \leq t \leq \mu$ implies $\left||\sigma_t(g_{xy})|_{r+t} - |g_{xy}|_r\right| \leq \epsilon$, where g_{xy} is the unique horizontal geodesic between x and y, and $|\,.\,|_r$ denotes the horizontal length with respect to m_r in the stratum $f^{-1}(r)$.

A forest-stack $\widetilde{X}$ equipped with a horizontal metric $\mathcal{H}$ will be denoted by $(\widetilde{X}, f, \sigma_t, \mathcal{H})$.

In other words, a horizontal metric on a forest-stack is a collection of metrics on the strata such that the length of the horizontal paths varies continuously when homotoping them along the orbits of the semi-flow. The definition of 'horizontal metric' does not imply that the horizontal distance varies continuously along the orbits. Figure 1 illustrates what might happen because of the possible non-injectivity of the maps $\sigma_t|_{f^{-1}(r)}$: if $\sigma_t(x) = \sigma_t(y)$ for two distinct points x, y in a horizontal geodesic $g \in f^{-1}(r)$ then $\sigma_t(g)$ is a horizontal path, but is not necessarily the image of an injective path. Thus the distance between the endpoints of $\sigma_t(g)$ is not realized by $\sigma_t(g)$ but by a path of smaller length, smaller at least than the length of $\sigma_t(g_{xy})$, where $g_{xy} \subset g$ is the subpath of g between x and y.

DEFINITION 3.4. Any horizontal geodesic g_{xy} between two distinct points x, y such that $\sigma_t(x) = \sigma_t(y)$ for some $t > 0$ is a *cancellation*.

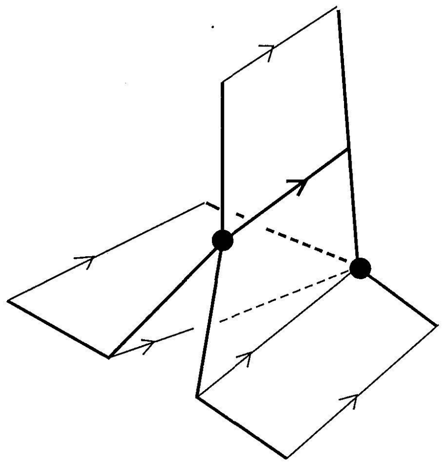

FIGURE 1
(A cancellation)

DEFINITION 3.5. Let p be a horizontal path in the stratum $f^{-1}(r)$ of a forest-stack $(\widetilde{X}, f, \sigma_t)$.

- The *pulled-tight projection* (or *image*) $[p]_{r+t}$ *of* p *on the stratum* $f^{-1}(r+t)$ is the unique horizontal geodesic between the endpoints of $\sigma_t(p)$ in the stratum $f^{-1}(r+t)$.
- A *geodesic preimage of* p *under* σ_t is any geodesic p_{-t} with $[p_{-t}]_{f(p_{-t})+t} = p$.

If S is a path in $\widetilde{X}$, the *pulled-tight projection of* S *on* $f^{-1}(r)$, $r \geq \max_{x \in S} f(x)$, is the unique horizontal geodesic which connects the images of the endpoints of S under the semi-flow in the stratum $f^{-1}(r)$.

3.2 TELESCOPIC METRIC

DEFINITION 3.6. A *telescopic path* in a forest-stack is a path which is the concatenation of non-degenerate horizontal and vertical subpaths.

The *vertical length of a telescopic path* p is equal to the sum of the vertical lengths of the maximal vertical subpaths of p.

If the considered forest-stack comes with a horizontal metric $\mathcal{H}$, the *horizontal length of a telescopic path* p is the sum of the horizontal lengths of the maximal horizontal subpaths of p.

The *telescopic length* $|p|_{(\widetilde{X},\mathcal{H})}$ of a telescopic path p in $\widetilde{X}$ is equal to the sum of the horizontal and vertical lengths of p.

We will always assume that our paths are equipped with an orientation, whatever it is, and we will denote by $i(p)$ (resp. $t(p)$) the initial (resp. terminal) point of a path p with respect to its orientation.

LEMMA-DEFINITION. *Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$. For any two points x, y in $\widetilde{X}$, we denote by $d_{(\widetilde{X},\mathcal{H})}(x, y)$ the infimum, over all the telescopic paths p in $\widetilde{X}$ between x and y, of their telescopic lengths $|p|_{(\widetilde{X},\mathcal{H})}$. Then $(\widetilde{X}, d_{(\widetilde{X},\mathcal{H})})$ is a $(1,2)$-quasi geodesic metric space. The map $d_{(\widetilde{X},\mathcal{H})} : \widetilde{X} \times \widetilde{X} \to \mathbf{R}^+$ is a* telescopic distance *associated to $\mathcal{H}$.*

Proof. If $d_{(\widetilde{X},\mathcal{H})}(x, y) = 0$ then $f(x) = f(y)$. The distance is realized as the infimum of the telescopic lengths of an infinite sequence $(T_n)_{n \in \mathbf{N}}$ of telescopic paths. There exists a unique horizontal geodesic between x and y. Otherwise any telescopic path between x and y has vertical length, and thus telescopic length, uniformly bounded away from zero. Let $\epsilon > 0$ be fixed. For some integer i all the telescopic paths $T_i, T_{i+1}, \ldots$ in the above sequence are contained in a box of height 2ϵ with horizontal boundaries the pulled-tight projection $[g]_{f(g)+\epsilon}$ and all the geodesic preimages of g under σ_ϵ. The vertical boundaries are the orbit-segments connecting the endpoints of the above geodesic preimages to the endpoints of $[g]_{f(g)+\epsilon}$. From the bounded-dilatation property, the horizontal length of each T_n for $n \geq i$ is at least $\lambda_+^{-2\epsilon} |[g]_{f(g)+\epsilon}|_{f(g)+\epsilon}$. Thus for any $n \geq i$, $|T_n|_{(\widetilde{X},\mathcal{H})} \geq \lambda_+^{-2\epsilon} |[g]_{f(g)+\epsilon}|_{f(g)+\epsilon}$. Since $\inf_{n \in \mathbf{N}} |T_n|_{(\widetilde{X},\mathcal{H})} = d_{(\widetilde{X},\mathcal{H})}(x, y) = 0$, we have $|[g]_{f(g)+\epsilon}|_{f(g)+\epsilon} = 0$.