3.2 Telescopic metric

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If S is a path in $\widetilde{X}$, the *pulled-tight projection of S on $f^{-1}(r)$*, $r \geq \max_{x \in S} f(x)$, is the unique horizontal geodesic which connects the images of the endpoints of S under the semi-flow in the stratum $f^{-1}(r)$.

3.2 TELESCOPIC METRIC

DEFINITION 3.6. A *telescopic path* in a forest-stack is a path which is the concatenation of non-degenerate horizontal and vertical subpaths.

The *vertical length of a telescopic path p* is equal to the sum of the vertical lengths of the maximal vertical subpaths of p.

If the considered forest-stack comes with a horizontal metric $\mathcal{H}$, the *horizontal length of a telescopic path p* is the sum of the horizontal lengths of the maximal horizontal subpaths of p.

The *telescopic length* $|p|_{(\widetilde{X},\mathcal{H})}$ of a telescopic path p in $\widetilde{X}$ is equal to the sum of the horizontal and vertical lengths of p.

We will always assume that our paths are equipped with an orientation, whatever it is, and we will denote by $i(p)$ (resp. $t(p)$) the initial (resp. terminal) point of a path p with respect to its orientation.

LEMMA-DEFINITION. *Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$. For any two points x, y in $\widetilde{X}$, we denote by $d_{(\widetilde{X},\mathcal{H})}(x,y)$ the infimum, over all the telescopic paths p in $\widetilde{X}$ between x and y, of their telescopic lengths $|p|_{(\widetilde{X},\mathcal{H})}$. Then $(\widetilde{X}, d_{(\widetilde{X},\mathcal{H})})$ is a $(1,2)$-quasi geodesic metric space. The map $d_{(\widetilde{X},\mathcal{H})} : \widetilde{X} \times \widetilde{X} \to \mathbf{R}^+$ is a* telescopic distance *associated to $\mathcal{H}$.*

Proof. If $d_{(\widetilde{X},\mathcal{H})}(x,y) = 0$ then $f(x) = f(y)$. The distance is realized as the infimum of the telescopic lengths of an infinite sequence $(T_n)_{n \in \mathbf{N}}$ of telescopic paths. There exists a unique horizontal geodesic between x and y. Otherwise any telescopic path between x and y has vertical length, and thus telescopic length, uniformly bounded away from zero. Let $\epsilon > 0$ be fixed. For some integer i all the telescopic paths $T_i, T_{i+1}, \ldots$ in the above sequence are contained in a box of height 2ϵ with horizontal boundaries the pulled-tight projection $[g]_{f(g)+\epsilon}$ and all the geodesic preimages of g under σ_ϵ. The vertical boundaries are the orbit-segments connecting the endpoints of the above geodesic preimages to the endpoints of $[g]_{f(g)+\epsilon}$. From the bounded-dilatation property, the horizontal length of each T_n for $n \geq i$ is at least $\lambda_+^{-2\epsilon} |[g]_{f(g)+\epsilon}|_{f(g)+\epsilon}$. Thus for any $n \geq i$, $|T_n|_{(\widetilde{X},\mathcal{H})} \geq \lambda_+^{-2\epsilon} |[g]_{f(g)+\epsilon}|_{f(g)+\epsilon}$. Since $\inf_{n \in \mathbf{N}} |T_n|_{(\widetilde{X},\mathcal{H})} = d_{(\widetilde{X},\mathcal{H})}(x,y) = 0$, we have $|[g]_{f(g)+\epsilon}|_{f(g)+\epsilon} = 0$.

That is, $\sigma_\epsilon(x) = \sigma_\epsilon(y)$. This holds for any $\epsilon > 0$. Since $(\sigma_t)_{t \in \mathbf{R}^+}$ depends continuously on t, we have $\sigma_0(x) = \sigma_0(y)$, whence $x = y$. We have thus proved that $d_{(\widetilde{X}, \mathcal{H})}$ does not vanish outside the diagonal of $\widetilde{X} \times \widetilde{X}$. The conclusion that this is a distance is now straightforward.

By definition of the telescopic distance, for any x, y in $\widetilde{X}$, for any $\epsilon > 0$, there exists a telescopic path p between x and y such that $|p|_{(\widetilde{X}, \mathcal{H})} \leq d_{(\widetilde{X}, \mathcal{H})}(x, y) + \epsilon$. We choose $\epsilon < \min(d_{(\widetilde{X}, \mathcal{H})}(x, y), 1)$. We consider the maximal collection of points $x_0, \ldots, x_k$ in p such that $x_0 = i(p)$, $x_k = t(p)$, and that the telescopic length of the subpath p_i of p between x_{i-1} and x_i is equal to ϵ for $i = 1, \ldots, k-1$. The maximality of the collection $\{x_0, x_1, \ldots, x_k\}$ implies that the telescopic length of the subpath p_k of p between x_{k-1} and x_k is at most ϵ. By definition $d_{(\widetilde{X}, \mathcal{H})}(x_{i-1}, x_i) \leq |p_i|_{(\widetilde{X}, \mathcal{H})}$ for $i = 1, \ldots, k$. Thus $d_{(\widetilde{X}, \mathcal{H})}(x_{i-1}, x_i) \leq 1$ for any $i = 1, \ldots, k$ and $\sum_{i=1}^{k} d_{(\widetilde{X}, \mathcal{H})}(x_{i-1}, x_i) \leq |p|_{(\widetilde{X}, \mathcal{H})}$. The choice of $\epsilon < d_{(\widetilde{X}, \mathcal{H})}(x, y)$ then implies that $\sum_{i=1}^{k} d_{(\widetilde{X}, \mathcal{H})}(x_{i-1}, x_i) \leq 2 d_{(\widetilde{X}, \mathcal{H})}(x, y)$. Therefore $x_0, x_1, \ldots, x_k$ is a $(1,2)$-quasi geodesic chain between x and y. □

REMARK 3.7. In nice cases, for instance in the case where the forest-stack is a proper metric space, the forest-stack is a true geodesic space.

4. MAIN THEOREM

DEFINITION 4.1. Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$.

1. The semi-flow is a *bounded-cancellation semi-flow* (with respect to $\mathcal{H}$) if there exist $\lambda_- \geq 1$ and $K \geq 0$ such that for any real $r \in \mathbf{R}$, for any horizontal geodesic $g \in f^{-1}(r)$, for any $t \geq 0$, $|[g]_{r+t}|_{r+t} \geq \lambda_-^{-t} |g|_r - K$.
2. The semi-flow is a *bounded-dilatation semi-flow* (with respect to $\mathcal{H}$) if there exists $\lambda_+ \geq 1$ such that for any real $r \in \mathbf{R}$, for any horizontal geodesic $g \in f^{-1}(r)$, for any $t \geq 0$, $|[g]_{r+t}|_{r+t} \leq \lambda_+^t |g|_r$.

REMARK 4.2. The reader can observe a dissymetry between the bounded-cancellation and bounded-dilatation properties, in the sense that the latter does not allow any additive constant. This is really necessary, since several proofs fail (e.g. those of Propositions 8.1 or 9.1) if an additive constant is allowed here.