# 5. PRELIMINARY WORK

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consider copies $\mathcal{R}_i$, $i = 0, 1, 2, \ldots$ of $\mathcal{R}$. We glue them to $\mathcal{R}$ as illustrated in Figure 2, that is by creating an infinite sequence of pockets of increasing size.

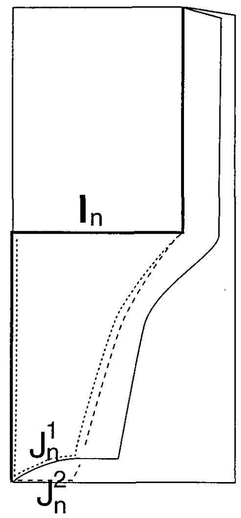


FIGURE 2
(A pocket)

We now attach copies of the negative half-plane of $\mathcal{R}$, along the horizontal lines with integer $y$-coordinate of the copies $\mathcal{R}_i$ of $\mathcal{R}$ considered above. In order to get a forest-stack whose strata are trees, we now identify a vertical half-line in each of the copies of the negative half-plane, ending at the horizontal line along which this copy was glued, to the corresponding vertical half-line in $\mathcal{R}$. In this way, we get a forest-stack whose strata are trees and whose semi-flow is as anounced. This forest-stack is not Gromov-hyperbolic because in each pocket (see Figure 2) the horizontal interval $I_n$ admits two preimages $J_n^1$, $J_n^2$ so that there are two telescopic geodesics joining the endpoints of $I_n$. These are the concatenation of $J_n^1$ and $J_n^2$ with the two vertical segments joining their endpoints to the endpoints of $I_n$. Since, by construction, there are pockets of arbitrarily large size, these two telescopic geodesics can be arbitrarily far from one another, so that the forest-stack is not Gromov-hyperbolic.

## 5. PRELIMINARY WORK

We consider a forest-stack $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ equipped with a horizontal metric $\mathcal{H}$ such that the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is strongly hyperbolic. Definition 4.3 introduces three *constants of hyperbolicity*, denoted by $\lambda$, $t_0$, $M$ in the sequel. The other constants of hyperbolicity, which appear in the bounded-dilatation and bounded-cancellation properties, are denoted by $\lambda_+$, $\lambda_-$, $K$. Any horizontal geodesic $g$ with horizontal length greater than $M$ satisfies at least one of the following two properties:

- The pulled-tight image $[g]_{nt_0}$ of $g$ after $nt_0$, $n \geq 1$, is $\lambda^n$ times longer than $g$. In this case the horizontal geodesic $g$ is *dilated in the future*, or more briefly *dilated, after* $t_0$.
- $g$ admits a geodesic preimage $g_{-nt_0}$ under $\sigma_{nt_0}$ which is $\lambda^n$ times longer than $g$. In this case, the horizontal geodesic $g$ is *dilated in the past after* $t_0$.

More generally, we will say that $g$ is *dilated in the future after* $kt_0$ (resp. *dilated in the past after* $kt_0$), $k \geq 1$, if the same inequalities hold only for any $n \geq k$, after replacing $\lambda^n$ by $\lambda^{(n+1-k)}$, and $g$ by $[g]_{r+(k-1)t_0}$ for the dilatation in the future and by $g_{-(k-1)t_0}$ for the dilatation in the past.

When the dilatation occurs in the past, only one geodesic preimage is required to have horizontal length $\lambda$ times the horizontal length of the horizontal geodesic $g$ considered. Thus it might happen, a priori, that the other geodesic preimages of $g$ remain short when returning to the past. Lemma 5.1 below shows that the constants of hyperbolicity can be chosen so that such a situation does not occur. This is a consequence of the bounded-cancellation property.

LEMMA 5.1. *Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack. Assume that $(\sigma_t)_{t \in \mathbf{R}^+}$ is (strongly) hyperbolic, with constants of hyperbolicity $\lambda$, $t_0$, $M$. Then,*

1) *There exist $t_0' = j\, t_0$, for some positive integer $j$, and $M' \geq M$ such that any horizontal geodesic $g \in f^{-1}(r)$ dilated in the past after $t_0'$, with $|g|_r \geq M'$, satisfies $\left|g_{-nt_0'}\right|_{r-nt_0'} \geq 2^n |g|_r$ for any geodesic preimage $g_{-nt_0'}$, $n \geq 1$.*

2) *The semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is (strongly) hyperbolic with constants of hyperbolicity $\lambda$, $t_0'$, $M'$, $\lambda_+'$, $\lambda_-'$, $K'$ for any $t_0' = j\, t_0$, $j \geq 1$ any positive integer, and any real numbers $M' \geq M$, $\lambda_+' \geq \lambda_+$, $\lambda_-' \geq \lambda_-$, $K' \geq K$. Furthermore, if the semi-flow satisfies (1) for some constants $t_0'$, $M'$, then it satisfies (1) for any $t_0'' = j\, t_0'$, where $j$ is any positive integer, and any real number $M'' \geq M'$.*

*Proof.* (2) is obvious. Let us check (1). We choose $t_0' \geq t_0$, $t_0' = jt_0$ with $j$ an integer, such that $\lambda^{t_0'} > 2$. We consider any horizontal geodesic

$g \in f^{-1}(r)$ with $|g|_r \geq M$. We assume that $g$ is dilated in the past after $t'_0$. Since the semi-flow is strongly hyperbolic, for each $n \geq 1$, in each connected component of $f^{-1}(r - nt'_0)$, there is at least one geodesic preimage $g_{-nt'_0}$ of $g$ with $|g_{-nt'_0}|_{r-nt'_0} \geq \lambda^{nt'_0}|g|_r$. We need an estimate of the horizontal length of the other geodesic preimages of $g$ in this stratum. Lemma 5.2 below is easily deduced from the bounded-cancellation property:

LEMMA 5.2. *With the assumptions and notation of Lemma 5.1, let $g \in f^{-1}(r)$ be some horizontal geodesic. If $g^1_{-t}$ and $g^2_{-t}$, $t > 0$, are two geodesic preimages of $g$ under $\sigma_t$ which belong to a same connected component of their stratum, then $\left||g^1_{-t}|_{r-t} - |g^2_{-t}|_{r-t}\right| \leq C_{5.2}(t)$ for some constant $C_{5.2}(t)$.*

Thus, by Lemma 5.2, for any $n \geq 1$, any geodesic preimage $g_{-nt'_0}$ satisfies $|g_{-nt'_0}|_{r-nt'_0} \geq \lambda^{nt'_0}|g|_r - C_{5.2}(nt'_0)$. For $n = 1$, if $|g|_r > \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2}$, then $|g_{-t'_0}|_{r-t'_0} > 2|g|_r$. Thus, if $|g|_r > \max(M, \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2})$ then any geodesic preimage $g_{-t'_0}$ has horizontal length greater than $2|g|_r$. In particular $|g_{-t'_0}|_{r-t'_0} \geq M$ because $|g|_r > M$. By definition of a hyperbolic semi-flow, $g_{-t'_0}$ is dilated either in the future or in the past. This cannot be the case in the future since $|g_{-t'_0}|_{r-t'_0} > |g|_r$. An easy induction on $n$ completes the proof. It suffices to set $t'_0 = (E[\max(1, \frac{\ln 2}{\ln \lambda})] + 1)t_0$ and $M' = \max(M, \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2}) + 1$. $\square$

We will assume that the constants of hyperbolicity $t_0$ and $M$ are chosen to satisfy the conclusion of Lemma 5.1. Moreover the constants of hyperbolicity $t_0$, $M$, $\lambda_+$, $\lambda_-$, $K$ are chosen large enough that computations make sense. In the sequel, we say that a path $g$ is *$C$-close* to a path $g'$ if $g$ and $g'$ are $C$-close with respect to the Hausdorff distance relative to the specified metric (the telescopic metric if none is specified). The indices of the constants refer to the lemma or proposition in which they first appear.

## 5.1 ABOUT DILATATION IN CANCELLATIONS

Let us recall that a *cancellation* is a horizontal geodesic whose endpoints are identified under some $\sigma_t$, $t > 0$.

LEMMA 5.3. *Let $g \in f^{-1}(r)$ be any horizontal geodesic which is dilated in the future after $nt_0$ for some integer $n \geq 1$. There exists a constant $C_{5.3}(n) \geq M$, which increases with $n$, such that if $g$ is contained in a cancellation, then $|g|_r \leq C_{5.3}(n)$.*

*Proof.* Let $c$ be the cancellation containing $g$. Let $c = c_1 \cup c_2$, with $[c_1]_{r+t} = [c_2]_{r+t}$ for some $t > 0$. We assume momentarily that $c_1 \cap c_2$ is an endpoint of $g$. The bounded-cancellation property implies that the horizontal length of a cancellation 'killed' in time $t_0$ (i.e. a cancellation whose pulled-tight projection after $t_0$ is a point) is a constant $C(t_0)$. This constant does not depend on the horizontal length of $g$.

Let us consider the pulled-tight image $[g]_{r+t_0}$. Let $p \subset [g]_{r+t_0}$ be the maximal subpath outside the pulled-tight image of $c$. This subpath $p$ is the image of a cancellation killed at time $t_0$. From the observation above and the bounded-dilatation property, $|p|_{r+t_0} \leq \lambda_+^{t_0} C(t_0)$. The same arguments lead to the upper bound $(\lambda_+^{nt_0} + \lambda_+^{(n-1)t_0} + \ldots + \lambda_+^{t_0})C(t_0)$ for the horizontal length of the subpath of $[g]_{r+nt_0}$ outside $[c]_{r+nt_0}$. Since $g$ is dilated in the future after $nt_0$, we have $|[g]_{r+nt_0}|_{r+nt_0} \geq \lambda^{t_0}|g|_r$. From the last two inequalities, if

$$|g|_r > \frac{(\lambda_+^{nt_0} + \lambda_+^{(n-1)t_0} + \ldots + \lambda_+^{t_0})C(t_0)}{\lambda^{t_0} - 1},$$

then the horizontal length of the subpath $q$ of $[g]_{r+nt_0}$ in $[c]_{r+nt_0}$ is greater than $|g|_r$. If $|g|_r \geq M$, then $|q|_{r+nt_0} \geq M$ is dilated in the future after $t_0$ since by convention $M$ satisfies the conclusion of Lemma 5.1. We thus obtain, for any $j \geq n$, the existence of a geodesic with horizontal length greater than $|g|_r$ in $[c]_{r+jt_0}$. This is impossible.

Let us now consider the case where $c_1 \cap c_2$ is not an endpoint of $g$. After some time $t > 0$, the situation will be the one described above, that is a cancellation $c' = c'_1 \cup c'_2$ with $c'_1 \cap c'_2$ an endpoint of $[g]_{r+t}$. The arguments above, together with the bounded-cancellation and bounded-dilatation properties, lead to the conclusion. □

We will often encounter situations in which the pulled-tight projection of a horizontal geodesic $p_1$ is identified with the pulled-tight projection of another horizontal geodesic $p_2$ in the same stratum. In this case $p_1, p_2$ are not necessarily contained in cancellations. But if they lie in the same connected component of their stratum, both are contained in the union of two cancellations. Lemma 5.4 below will allow us to deal with similar situations.

LEMMA 5.4. *Let $p$ be a horizontal geodesic which admits a decomposition in $r$ subpaths $p_i$ such that for some constant $L \geq 0$, for any $i = 1, \ldots, r$, either $|[p_i]_{r+nt_0}|_{r+nt_0} \leq |p_i|_r$ or $L \geq |[p_i]_{r+nt_0}|_{r+nt_0} > |p_i|_r$. Then there exists a constant $C_{5.4}(n, r, L)$, which is increasing in each variable, such that if $p$ is dilated in the future after $nt_0$, then $|p|_r \leq C_{5.4}(n, r, L)$.*

*Proof.* We set $n = 1$ in order to simplify the notation; the general case is treated in the same way. Up to permuting the indices, $|[p_i]_{r+t_0}|_{r+t_0} > |p_i|_r$ for $i = 1, \ldots, j$. Since $p$ is dilated in the future after $t_0$,

$$jL + \sum_{i=j+1}^{r} |p_i|_r \geq \lambda^{t_0} \sum_{i=1}^{r} |p_i|_r .$$

Therefore $|p|_r \leq \frac{jL}{\lambda^{t_0} - 1}$. □

## 5.2 STRAIGHT TELESCOPIC PATHS

DEFINITION 5.5. A *straight* telescopic path is a telescopic path $S$ such that if $x$, $y$ are any two points in $S$ with $x \in O^+(y) \cup O^-(y)$ then the subpath of $S$ between $x$ and $y$ is equal to the orbit-segment of the semi-flow between $x$ and $y$.

If $S$ is a path containing a point $x$, let $S_{x,t} \subset S$ be the maximal subpath of $S$ containing $x$, whose pulled-tight projection $[S_{x,t}]_{f(x)+t}$ on $f^{-1}(f(x)+t)$ is well defined. The point $\sigma_t(x)$ does not necessarily belong to $[S_{x,t}]_{f(x)+t}$. However there exists a unique point in $[S_{x,t}]_{f(x)+t}$ which minimizes the horizontal distance between $\sigma_t(x)$ and $[S_{x,t}]_{f(x)+t}$. This point is denoted by $\overline{x}_t$. Lemma 5.6 below gives an upper bound, depending on $t$, for the telescopic distance between $x$ and $\overline{x}_t$.

LEMMA 5.6. *Let $S$ be any straight telescopic path. If $t$ is any non negative real number, there exists a constant $C_{5.6}(t) \geq t$, which increases with $t$, such that any point $x \in S$ is at telescopic distance smaller than $C_{5.6}(t)$ from the point $\overline{x}_t$ (see above).*

*Proof.* If $\sigma_t(x) \in [S_{x,t}]_{f(x)+t}$, we set $C_{5.6}(t) = t$. Since $S$ is straight, if $\sigma_t(x) \notin [S_{x,t}]_{f(x)+t}$, $x$ belongs to a cancellation $c$ whose endpoints lie in the past orbits of $\overline{x}_t$. The bounded-cancellation property gives an upper bound on the horizontal length of $c$. This leads to the conclusion. □