6. About straight quasi geodesics

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

6. About straight quasi geodesics

DEFINITION 6.1. Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack. A *(J,J')-quasi geodesic*, $J \geq 1$, $J' \geq 0$, in $(\widetilde{X}, d_{(\widetilde{X},\mathcal{H})})$ is a telescopic path S of which each subpath S' satisfies the inequality

$$|S'|_{(\widetilde{X},\mathcal{H})} \leq J d_{(\widetilde{X},\mathcal{H})}(i(S'), t(S')) + J' .$$

LEMMA 6.2. *Let p be a straight (J,J')-quasi geodesic with*

$$|r_{max} - f(i(p))| \leq t_0 ,$$

where $r_{max} = max_{x \in p} f(x)$. There exists a constant $C_{6.2}(J,J') \geq M$, which increases with J and J', such that if $|[p]_{r_{max}}|_{r_{max}} \geq C_{6.2}(J,J')$ then $[p]_{r_{max}}$ is dilated both in the future and in the past after $C_{6.2}(J,J')t_0$.

Proof. By the bounded-dilatation property, $|p|_{(\widetilde{X},\mathcal{H})} \geq \lambda_+^{-t_0}|[p]_{r_{max}}|_{r_{max}} + t_0$. We choose n_* so that $\lambda_+^{-t_0} - J\lambda^{-n_* t_0} > 0$. For any n greater than n_*, the inequality

$$J(2t_0 + 2nt_0 + \lambda^{-nt_0}|[p]_{r_{max}}|_{r_{max}}) + J' < \lambda_+^{-t_0}|[p]_{r_{max}}|_{r_{max}} + t_0$$

is satisfied for $|[p]_{r_{max}}|_{r_{max}} > \frac{(2J-1)t_0 + 2nJt_0 + J'}{\lambda_+^{-t_0} - J\lambda^{-nt_0}}$. This is in contradiction with p being a (J,J')-quasi geodesic. If $|[p]_{r_{max}}|_{r_{max}} > \lambda_+^{n_* t_0} M$, then, by the bounded-dilatation property, the geodesic preimages of $[p]_{r_{max}}$ under $\sigma_{n_* t_0}$ have horizontal length at least M. Hence, if moreover $|[p]_{r_{max}}|_{r_{max}} > \frac{(2J-1)t_0 + 2n_* Jt_0 + J'}{\lambda_+^{-t_0} - J\lambda^{-n_* t_0}}$ then the hyperbolicity of the semi-flow implies that they are dilated in the past after t_0. The bounded-dilatation property implies that these geodesic preimages have horizontal length at least $\lambda_+^{-n_* t_0}|[p]_{r_{max}}|_{r_{max}}$. Choosing N_* such that $\lambda^{N_* t_0} \geq \lambda_+^{n_* t_0}$, we conclude that $[p]_{r_{max}}$ is dilated in the past after $(N_* + 1)t_0$. The same arguments allow us to find a lower bound on $|[p]_{r_{max}}|_{r_{max}}$ for $[p]_{r_{max}}$ to be dilated in the future after some fixed finite time. □

DEFINITION 6.3. Let $(\widetilde{X}, f, \sigma_t)$ be a forest-stack. A *stair* in $\widetilde{X}$ is a telescopic path along which the function f is monotone.

LEMMA 6.4. *Let p be a straight (J,J')-quasi geodesic stair between two points a and b, $f(a) \leq f(b)$. There exists a constant $C_{6.4}(J,J') \geq M$, which increases with J and J', such that if the horizontal length of a horizontal geodesic I between a and $O^-(b)$ (resp. b and $O^+(a)$) is at least $C_{6.4}(J,J')$, then I is dilated in the past (resp. in the future) after t_0.*

Proof. Let X be such that $\lambda^{t_0}X > X + \lambda_+^{t_0}C_{6.2}(J,J')$. Assume that the horizontal length of some horizontal geodesic I between a and $O^-(b)$ is at least X. By Lemma 6.2, the choice of X implies that if I is dilated in the future after t_0, then the first point a_1 along p satisfying $f(a_1) = f(a) + t_0$ is at horizontal distance greater than X from $O^-(b)$. By induction, we thus obtain an infinite sequence of points $a_1, a_2, \dots, a_n, \dots$ in p such that $f(a_i) = f(a_{i-1}) + t_0$ and each a_i is at horizontal distance at least X from $O^-(b)$. This is absurd. The other case of Lemma 6.4 is treated similarly. □

DEFINITION 6.5. Let S_0, S_1 be two telescopic paths whose pulled-tight projections agree after some finite time. We say that S_0 and S_1 *are in fine position* if, for any two points x, y, $x \neq y$, satisfying $x \in S_i \cap O(y)$, $y \in S_{i+1}$, $i = 0, 1 \mod 2$, then $x \in O^+(y) \cup O^-(y)$.

Let us observe that a path is always in fine position with respect to any of its pulled-tight projections.

DEFINITION 6.6. A $+$*-hole* (resp. $-$*-hole*) is a telescopic path with both endpoints in a same stratum, which is in fine position with respect to the horizontal geodesic I between its endpoints, and which satisfies furthermore $\min_{x \in p} f(x) \geq f(I)$ (resp. $\max_{x \in p} f(x) \leq f(I)$).

LEMMA 6.7. *Let p be a straight (J,J')-quasi geodesic $+$-hole (resp. $-$-hole). There exists a constant $C_{6.7}(J,J') \geq M$, which increases with J and J', such that, if I is the horizontal geodesic between the endpoints of p and if $|I|_{f(I)} \geq C_{6.7}(J,J')$, then I is dilated in the past (resp. future) after $C_{6.7}(J,J')t_0$.*

Proof. We consider a decomposition $p_1 p_2 \dots p_l$ of p such that

$$\max_{x \in p_i} |f(x) - f(i(p_i))| \leq t_0 ,$$

and a decomposition $I_1 \dots I_l$ of I, where I_k joins the past orbits of the endpoints of p_k. We denote by I_D the union of the I_k's which are dilated in the past after $C_{6.2}(J,J')t_0$, and by I_C the union of the other intervals in I. By Lemma 6.2, the horizontal length of any interval in I_C is at most $C_{6.2}(J,J')$.

Let n be some positive integer. We consider a horizontal geodesic h with $I = [h]_{f(h)+nC_{6.2}(J,J')t_0}$ and assume that h is dilated in the future after t_0. Then,

$$\lambda^n |I_D|_{f(I)} + \lambda_+^{-n} |I_C|_{f(I)} \leq |h|_{f(h)} \leq \lambda^{-n}(|I_D|_{f(I)} + |I_C|_{f(I)}) .$$

Hence $|I_C|_{f(I)} \geq \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda^{-n}} |I_D|_{f(I)}$, so that $|I_C|_{f(I)} \geq \frac{X(n)}{1+X(n)} |I|_{f(I)}$ with $X(n) = \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda^{-n}}$. Now $\lim_{n \to +\infty} \frac{X(n)}{1+X(n)} = 1$, so that for some $n_* \geq 1$, for any $n \geq n_*$, $\frac{X(n)}{1+X(n)} \geq \frac{1}{2}$. Since the horizontal length of any interval I_k in I_C is at most $C_{6.2}(J, J')$, and the telescopic length of the associated $p_k \subset p$ is at least t_0, we obtain

$$|p|_{(\widetilde{X}, \mathcal{H})} \geq \frac{t_0}{2C_{6.2}(J, J')} |I|_{f(I)}.$$

On the other hand, $|p|_{(\widetilde{X}, \mathcal{H})} \leq 2Jnt_0 + \lambda^{-n} J |I|_{f(I)} + J'$ for any $n \geq n_*$. The last two inequalities give, for $n \geq n_*$, $2Jnt_0 + \lambda^{-n} J |I|_{f(I)} + J' \geq \frac{t_0}{2C_{6.2}(J,J')} |I|_{f(I)}$, equivalently $2Jnt_0 + J' \geq (\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n} J) |I|_{f(I)}$. We choose $n_\circ \geq n_*$ such that $\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J > 0$. We get

$$\frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J} \geq |I|_{f(I)}.$$

Thus, for $|I|_{f(I)} > \frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J}$, h is not dilated in the future after t_0. If $|I|_{f(I)} > \lambda_+^{n_\circ} M$, then $|h|_{f(h)} \geq M$. Therefore h is dilated in the past after t_0. We choose N such that $\lambda^N \lambda_+^{-n_\circ} > \lambda$. Thus, if $|I|_{f(I)} \geq \max(\lambda_+^{n_\circ} M, \frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J})$ then I is dilated in the past after $(n_\circ C_{6.2}(J, J') + N) t_0$. The arguments and computations in the case where $\max_{x \in p} f(x) \leq f(I)$ are the same. □

7. Substitution of quasi geodesics

LEMMA 7.1. *Let p be a (J, J')-quasi geodesic. Let q be obtained from p by replacing subpaths $p_i \subset p$ by (L, L')-quasi geodesics q_i satisfying the following properties:*

- *q_i has the same endpoints as p_i,*
- *q_i is L-close to p_i,*
- *$|q_i|_{(\widetilde{X}, \mathcal{H})} \leq L |p_i|_{(\widetilde{X}, \mathcal{H})}$.*

There exists a constant $C_{7.1}(L, L', J, J')$, which increases in each variable, such that q is a $(C_{7.1}(L, L', J, J'), C_{7.1}(L, L', J, J'))$-quasi geodesic which is L-close to p.

Proof. Since each q_i is L-close to a p_i, and with the same endpoints, q is L-close to p. Let us consider any two points x, y in q and let $q_{xy} \subset q$