

## 6. About straight quasi geodesics

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6. ABOUT STRAIGHT QUASI GEODESICS

DEFINITION 6.1. Let  $(\tilde{X}, f, \sigma_t, \mathcal{H})$  be a forest-stack. A  $(J, J')$ -quasi geodesic,  $J \geq 1, J' \geq 0$ , in  $(\tilde{X}, d_{(\tilde{X}, \mathcal{H})})$  is a telescopic path  $S$  of which each subpath  $S'$  satisfies the inequality

$$|S'|_{(\tilde{X}, \mathcal{H})} \leq Jd_{(\tilde{X}, \mathcal{H})}(i(S'), t(S')) + J'.$$

LEMMA 6.2. Let  $p$  be a straight  $(J, J')$ -quasi geodesic with

$$|r_{max} - f(i(p))| \leq t_0,$$

where  $r_{max} = \max_{x \in p} f(x)$ . There exists a constant  $C_{6.2}(J, J') \geq M$ , which increases with  $J$  and  $J'$ , such that if  $|[p]_{r_{max}}|_{r_{max}} \geq C_{6.2}(J, J')$  then  $[p]_{r_{max}}$  is dilated both in the future and in the past after  $C_{6.2}(J, J')t_0$ .

*Proof.* By the bounded-dilatation property,  $|p|_{(\tilde{X}, \mathcal{H})} \geq \lambda_+^{-t_0} |[p]_{r_{max}}|_{r_{max}} + t_0$ . We choose  $n_*$  so that  $\lambda_+^{-t_0} - J\lambda^{-n_*t_0} > 0$ . For any  $n$  greater than  $n_*$ , the inequality

$$J(2t_0 + 2nt_0 + \lambda^{-nt_0} |[p]_{r_{max}}|_{r_{max}}) + J' < \lambda_+^{-t_0} |[p]_{r_{max}}|_{r_{max}} + t_0$$

is satisfied for  $|[p]_{r_{max}}|_{r_{max}} > \frac{(2J-1)t_0 + 2nJt_0 + J'}{\lambda_+^{-t_0} - J\lambda^{-nt_0}}$ . This is in contradiction with  $p$  being a  $(J, J')$ -quasi geodesic. If  $|[p]_{r_{max}}|_{r_{max}} > \lambda_+^{n_*t_0} M$ , then, by the bounded-dilatation property, the geodesic preimages of  $[p]_{r_{max}}$  under  $\sigma_{n_*t_0}$  have horizontal length at least  $M$ . Hence, if moreover  $|[p]_{r_{max}}|_{r_{max}} > \frac{(2J-1)t_0 + 2n_*Jt_0 + J'}{\lambda_+^{-t_0} - J\lambda^{-n_*t_0}}$  then the hyperbolicity of the semi-flow implies that they are dilated in the past after  $t_0$ . The bounded-dilatation property implies that these geodesic preimages have horizontal length at least  $\lambda_+^{-n_*t_0} |[p]_{r_{max}}|_{r_{max}}$ . Choosing  $N_*$  such that  $\lambda^{N_*t_0} \geq \lambda_+^{n_*t_0}$ , we conclude that  $[p]_{r_{max}}$  is dilated in the past after  $(N_* + 1)t_0$ . The same arguments allow us to find a lower bound on  $|[p]_{r_{max}}|_{r_{max}}$  for  $[p]_{r_{max}}$  to be dilated in the future after some fixed finite time.  $\square$

DEFINITION 6.3. Let  $(\tilde{X}, f, \sigma_t)$  be a forest-stack. A *stair* in  $\tilde{X}$  is a telescopic path along which the function  $f$  is monotone.

LEMMA 6.4. Let  $p$  be a straight  $(J, J')$ -quasi geodesic stair between two points  $a$  and  $b$ ,  $f(a) \leq f(b)$ . There exists a constant  $C_{6.4}(J, J') \geq M$ , which increases with  $J$  and  $J'$ , such that if the horizontal length of a horizontal geodesic  $I$  between  $a$  and  $O^-(b)$  (resp.  $b$  and  $O^+(a)$ ) is at least  $C_{6.4}(J, J')$ , then  $I$  is dilated in the past (resp. in the future) after  $t_0$ .

*Proof.* Let  $X$  be such that  $\lambda^{t_0} X > X + \lambda_+^{t_0} C_{6.2}(J, J')$ . Assume that the horizontal length of some horizontal geodesic  $I$  between  $a$  and  $O^-(b)$  is at least  $X$ . By Lemma 6.2, the choice of  $X$  implies that if  $I$  is dilated in the future after  $t_0$ , then the first point  $a_1$  along  $p$  satisfying  $f(a_1) = f(a) + t_0$  is at horizontal distance greater than  $X$  from  $O^-(b)$ . By induction, we thus obtain an infinite sequence of points  $a_1, a_2, \dots, a_n, \dots$  in  $p$  such that  $f(a_i) = f(a_{i-1}) + t_0$  and each  $a_i$  is at horizontal distance at least  $X$  from  $O^-(b)$ . This is absurd. The other case of Lemma 6.4 is treated similarly.  $\square$

DEFINITION 6.5. Let  $S_0, S_1$  be two telescopic paths whose pulled-tight projections agree after some finite time. We say that  $S_0$  and  $S_1$  are in fine position if, for any two points  $x, y, x \neq y$ , satisfying  $x \in S_i \cap O(y), y \in S_{i+1}, i = 0, 1 \pmod{2}$ , then  $x \in O^+(y) \cup O^-(y)$ .

Let us observe that a path is always in fine position with respect to any of its pulled-tight projections.

DEFINITION 6.6. A  $+hole$  (resp.  $-hole$ ) is a telescopic path with both endpoints in a same stratum, which is in fine position with respect to the horizontal geodesic  $I$  between its endpoints, and which satisfies furthermore  $\min_{x \in p} f(x) \geq f(I)$  (resp.  $\max_{x \in p} f(x) \leq f(I)$ ).

LEMMA 6.7. Let  $p$  be a straight  $(J, J')$ -quasi geodesic  $+hole$  (resp.  $-hole$ ). There exists a constant  $C_{6.7}(J, J') \geq M$ , which increases with  $J$  and  $J'$ , such that, if  $I$  is the horizontal geodesic between the endpoints of  $p$  and if  $|I|_{f(I)} \geq C_{6.7}(J, J')$ , then  $I$  is dilated in the past (resp. future) after  $C_{6.7}(J, J')t_0$ .

*Proof.* We consider a decomposition  $p_1 p_2 \dots p_l$  of  $p$  such that

$$\max_{x \in p_i} |f(x) - f(i(p_i))| \leq t_0,$$

and a decomposition  $I_1 \dots I_l$  of  $I$ , where  $I_k$  joins the past orbits of the endpoints of  $p_k$ . We denote by  $I_D$  the union of the  $I_k$ 's which are dilated in the past after  $C_{6.2}(J, J')t_0$ , and by  $I_C$  the union of the other intervals in  $I$ . By Lemma 6.2, the horizontal length of any interval in  $I_C$  is at most  $C_{6.2}(J, J')$ .

Let  $n$  be some positive integer. We consider a horizontal geodesic  $h$  with  $I = [h]_{f(h) + nC_{6.2}(J, J')t_0}$  and assume that  $h$  is dilated in the future after  $t_0$ . Then,

$$\lambda^n |I_D|_{f(I)} + \lambda_+^{-n} |I_C|_{f(I)} \leq |h|_{f(h)} \leq \lambda^{-n} (|I_D|_{f(I)} + |I_C|_{f(I)}).$$

Hence  $|I_C|_{f(I)} \geq \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda_-^{-n}} |I_D|_{f(I)}$ , so that  $|I_C|_{f(I)} \geq \frac{X(n)}{1+X(n)} |I|_{f(I)}$  with  $X(n) = \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda_-^{-n}}$ . Now  $\lim_{n \rightarrow +\infty} \frac{X(n)}{1+X(n)} = 1$ , so that for some  $n_* \geq 1$ , for any  $n \geq n_*$ ,  $\frac{X(n)}{1+X(n)} \geq \frac{1}{2}$ . Since the horizontal length of any interval  $I_k$  in  $I_C$  is at most  $C_{6.2}(J, J')$ , and the telescopic length of the associated  $p_k \subset p$  is at least  $t_0$ , we obtain

$$|p|_{(\tilde{X}, \mathcal{H})} \geq \frac{t_0}{2C_{6.2}(J, J')} |I|_{f(I)}.$$

On the other hand,  $|p|_{(\tilde{X}, \mathcal{H})} \leq 2Jnt_0 + \lambda^{-n}J |I|_{f(I)} + J'$  for any  $n \geq n_*$ . The last two inequalities give, for  $n \geq n_*$ ,  $2Jnt_0 + \lambda^{-n}J |I|_{f(I)} + J' \geq \frac{t_0}{2C_{6.2}(J, J')} |I|_{f(I)}$ , equivalently  $2Jnt_0 + J' \geq (\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n}J) |I|_{f(I)}$ . We choose  $n_o \geq n_*$  such that  $\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J > 0$ . We get

$$\frac{2Jn_o t_0 + J'}{\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J} \geq |I|_{f(I)}.$$

Thus, for  $|I|_{f(I)} > \frac{2Jn_o t_0 + J'}{\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J}$ ,  $h$  is not dilated in the future after  $t_0$ . If  $|I|_{f(I)} > \lambda_+^{n_o}M$ , then  $|h|_{f(h)} \geq M$ . Therefore  $h$  is dilated in the past after  $t_0$ . We choose  $N$  such that  $\lambda^N \lambda_+^{-n_o} > \lambda$ . Thus, if  $|I|_{f(I)} \geq \max(\lambda_+^{n_o}M, \frac{2Jn_o t_0 + J'}{\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J})$  then  $I$  is dilated in the past after  $(n_o C_{6.2}(J, J') + N)t_0$ . The arguments and computations in the case where  $\max_{x \in p} f(x) \leq f(I)$  are the same.  $\square$

### 7. SUBSTITUTION OF QUASI GEODESICS

LEMMA 7.1. *Let  $p$  be a  $(J, J')$ -quasi geodesic. Let  $q$  be obtained from  $p$  by replacing subpaths  $p_i \subset p$  by  $(L, L')$ -quasi geodesics  $q_i$  satisfying the following properties:*

- $q_i$  has the same endpoints as  $p_i$ ,
- $q_i$  is  $L$ -close to  $p_i$ ,
- $|q_i|_{(\tilde{X}, \mathcal{H})} \leq L|p_i|_{(\tilde{X}, \mathcal{H})}$ .

*There exists a constant  $C_{7.1}(L, L', J, J')$ , which increases in each variable, such that  $q$  is a  $(C_{7.1}(L, L', J, J'), C_{7.1}(L, L', J, J'))$ -quasi geodesic which is  $L$ -close to  $p$ .*

*Proof.* Since each  $q_i$  is  $L$ -close to a  $p_i$ , and with the same endpoints,  $q$  is  $L$ -close to  $p$ . Let us consider any two points  $x, y$  in  $q$  and let  $q_{xy} \subset q$