Zeitschrift: L'Enseignement Mathématique

Band: 49 (2003)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HYPERBOLICITY OF MAPPING-TORUS GROUPS AND SPACES

Kapitel: 7. Substitution of quasi geodesics

Autor: Gautero, François

DOI: https://doi.org/10.5169/seals-66690

Hence $|I_C|_{f(I)} \geq \frac{\lambda^n-\lambda_+^{-n}}{\lambda^n-\lambda^{-n}}|I_D|_{f(I)}$, so that $|I_C|_{f(I)} \geq \frac{X(n)}{1+X(n)}|I|_{f(I)}$ with $X(n) = \frac{\lambda^n-\lambda_+^{-n}}{\lambda^n-\lambda^{-n}}$. Now $\lim_{n\to+\infty} \frac{X(n)}{1+X(n)} = 1$, so that for some $n_* \geq 1$, for any $n \geq n_*$, $\frac{X(n)}{1+X(n)} \geq \frac{1}{2}$. Since the horizontal length of any interval I_k in I_C is at most $C_{6.2}(J,J')$, and the telescopic length of the associated $p_k \subset p$ is at least t_0, we obtain

$$|p|_{(\widetilde{X},\mathcal{H})} \geq \frac{t_0}{2C_{6.2}(J,J')}|I|_{f(I)}.$$

On the other hand, $|p|_{(\widetilde{X},\mathcal{H})} \leq 2Jnt_0 + \lambda^{-n}J|I|_{f(I)} + J'$ for any $n \geq n_*$. The last two inequalities give, for $n \geq n_*$, $2Jnt_0 + \lambda^{-n}J|I|_{f(I)} + J' \geq \frac{t_0}{2C_{6.2}(J,J')}|I|_{f(I)}$, equivalently $2Jnt_0 + J' \geq (\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n}J)|I|_{f(I)}$. We choose $n_\circ \geq n_*$ such that $\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ}J > 0$. We get

$$\frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ}J} \geq |I|_{f(I)}.$$

Thus, for $|I|_{f(I)} > \frac{2Jn_\circ t_0+J'}{\frac{t_0}{2C_{6.2}(J,J')}-\lambda^{-n_\circ}J}$, h is not dilated in the future after t_0. If $|I|_{f(I)} > \lambda_+^{n_\circ}M$, then $|h|_{f(h)} \geq M$. Therefore h is dilated in the past after t_0. We choose N such that $\lambda^N\lambda_+^{-n_\circ} > \lambda$. Thus, if $|I|_{f(I)} \geq \max(\lambda_+^{n_\circ}M, \frac{2Jn_\circ t_0+J'}{\frac{t_0}{2C_{6.2}(J,J')}-\lambda^{-n_\circ}J})$ then I is dilated in the past after $(n_\circ C_{6.2}(J,J') + N)t_0$. The arguments and computations in the case where $\max_{x\in p} f(x) \leq f(I)$ are the same. □

7. Substitution of quasi geodesics

LEMMA 7.1. *Let p be a (J,J')-quasi geodesic. Let q be obtained from p by replacing subpaths $p_i \subset p$ by (L,L')-quasi geodesics q_i satisfying the following properties:*

- *q_i has the same endpoints as p_i,*
- *q_i is L-close to p_i,*
- *$|q_i|_{(\widetilde{X},\mathcal{H})} \leq L|p_i|_{(\widetilde{X},\mathcal{H})}$.*

There exists a constant $C_{7.1}(L,L',J,J')$, which increases in each variable, such that q is a $(C_{7.1}(L,L',J,J'), C_{7.1}(L,L',J,J'))$-quasi geodesic which is L-close to p.

Proof. Since each q_i is L-close to a p_i, and with the same endpoints, q is L-close to p. Let us consider any two points x, y in q and let $q_{xy} \subset q$

be the subpath of q between x and y. If both x and y lie in a q_i, or in a same subpath in the closed complement of the union of the q_i's, then $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq \max(L,J)d_{(\widetilde{X},\mathcal{H})}(x,y) + \max(L',J')$. Otherwise $q_{xy} = w_1w_2w_3$, where w_1, w_3 are contained either in some q_i or in p, and w_2 begins and ends with the initial or terminal point of some q_i. The third property concerning the q_i's leads to $|w_2|_{(\widetilde{X},\mathcal{H})} \leq L|p_2|_{(\widetilde{X},\mathcal{H})}$, where $p_2 \subset p$ is the subpath of p with the same endpoints as w_2. Thus $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq LJd_{(\widetilde{X},\mathcal{H})}(x,y) + 2\max(L',LJ')$. □

LEMMA 7.2. *Let p be a straight (J,J')-quasi geodesic $-$-hole such that $\max_{x\in p} f(I) - f(x) \leq L$, where I is the horizontal geodesic joining the endpoints of p. Then there exists a constant $C_{7.2}(L,J,J') \geq M$, which increases in each variable, such that*

1) $|I|_{f(I)} \leq C_{7.2}(L,J,J')|p|_{(\widetilde{X},\mathcal{H})}$.

2) *I is a straight $(C_{7.2}(L,J,J'), C_{7.2}(L,J,J'))$-quasi geodesic which is $C_{7.2}(L,J,J')$-close to p.*

Proof. A horizontal geodesic is always straight. The horizontal geodesic I is the pulled-tight projection of p. Thus, by the bounded-dilatation property, $|I|_{f(I)} \leq \lambda_+^L |p|_{(\widetilde{X},\mathcal{H})}$. By Lemma 5.6, I is $C_{5.6}(L)$-close to p. Consider any subpath I' of I; it is the pulled-tight projection of some subpath p' of p. By the bounded-dilatation property, $|I'|_{f(I)} \leq \lambda_+^L |p'|_{(\widetilde{X},\mathcal{H})}$. Since p is a (J,J')-quasi geodesic, $|I'|_{f(I)} \leq \lambda_+^L (Jd_{(\widetilde{X},\mathcal{H})}(i(p'),t(p')) + J')$. Since I' is $C_{5.6}(L)$-close to p', $|I'|_{f(I)} \leq \lambda_+^L Jd_{(\widetilde{X},\mathcal{H})}(i(I'),t(I')) + \lambda_+^L(2JC_{5.6}(L) + J')$. □

LEMMA 7.3. *Let p be a straight (J,J')-quasi geodesic $-$-hole such that the horizontal length of the horizontal geodesic I between its endpoints is less than or equal to L. Then there exists a constant $C_{7.3}(L,J,J') \geq M$, which increases in each variable, such that*

1) $|I|_{f(I)} \leq C_{7.3}(L,J,J')|p|_{(\widetilde{X},\mathcal{H})}$.

2) *I is a straight $(C_{7.3}(L,J,J'), C_{7.3}(L,J,J'))$-quasi geodesic which is $C_{7.3}(L,J,J')$-close to p.*

Proof. Since p is a (J,J')-quasi geodesic,

$$\max_{x\in p} |f(x) - f(I)| \leq J|I|_{f(I)} + J' .$$

Lemma 7.3 now follows from Lemma 7.2. □

LEMMA 7.4. *Let p be a straight (J,J')-quasi geodesic stair. For any $L \geq 0$, there exists a constant $C_{7.4}(L,J,J')$, which increases in each variable, such that if q is a straight stair whose points are at horizontal distance at most L from p, and with the same endpoints as p, then*

1) *q is a straight $(C_{7.4}(L,J,J'), C_{7.4}(L,J,J'))$-quasi geodesic stair which is L-close to p.*

2) $|q|_{(\widetilde{X},\mathcal{H})} \leq C_{7.4}(L,J,J')|p|_{(\widetilde{X},\mathcal{H})}$.

Proof. Consider a stair S, in the disc bounded by $p \cup q$, whose endpoints are those of p and q, and whose vertical geodesics end at q, all the stairs being oriented so that f is increasing along them. Consider a subpath S' of S which is the concatenation of a vertical segment followed by a horizontal one. By assumption, the horizontal length X of S' is bounded above by L. Let t be its vertical length. The bounded-dilatation property implies that the quotient of $|S'|_{(\widetilde{X},\mathcal{H})}$ by the telescopic length of the subpath of p between the endpoints of S' is bounded above by $Q = \frac{t+X}{t+\lambda_+^{-t}X}$. Since $X \leq L$, Q tends to 1 as $t \to +\infty$. One thus obtains a constant T such that for $t \geq T$, Q is bounded above by some constant, depending on L. When both t and X are close to 0 then Q is close to 1. Hence, since Q is continuous, Q admits an upper bound, denoted by $A(L)$, for all the t and X considered. This upper bound will be the same for all the subpaths S' as above.

The stair S is a concatenation of such subpaths S', possibly with one or two subpaths of p at the extremities. Thus the additivity of the telescopic length gives $|S|_{(\widetilde{X},\mathcal{H})} \leq A(L)|p|_{(\widetilde{X},\mathcal{H})}$. Let S'' be a subpath of S which is the concatenation of a horizontal subpath followed by a vertical one. The path S is the concatenation of such subpaths S'' with possibly one or two subpaths of q at the extremities. Exactly the same arguments as above give $|q|_{(\widetilde{X},\mathcal{H})} \leq A(L)|S|_{(\widetilde{X},\mathcal{H})}$. We thus get $|q|_{(\widetilde{X},\mathcal{H})} \leq A(L)^2|p|_{(\widetilde{X},\mathcal{H})}$. It only remains to prove that q is a quasi geodesic with constants of quasi geodesicity depending only on L, J, J'. Let x, y be any two points in q. As usual q_{xy} is the subpath of q between x and y and we denote by $p_{x'y'}$ the subpath of p between the two points x', y' in p which are at horizontal distance at most L from x and y. We consider a stair S between q_{xy} and $p_{x'y'}$, with the same endpoints as q_{xy}. The same arguments as above apply and give $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq A(L)^2|p_{x'y'}|_{(\widetilde{X},\mathcal{H})}$. Since p is a (J,J')-quasi geodesic, we conclude that $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq JA(L)^2 d_{(\widetilde{X},\mathcal{H})}(x',y') + J'A(L)^2$. Since $d_{(\widetilde{X},\mathcal{H})}(x',y') \leq d_{(\widetilde{X},\mathcal{H})}(x,y) + 2L$, the proof of Lemma 7.4 is complete. □