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Thus $a_I\omega_1 b_I b_I^{-1}\omega_1^{-1}\omega^{-1}a_I^{-1} \in \mathrm{Im}(\alpha)$, $a'_I\omega_1 b'_I b'^{-1}_I\omega_1^{-1}\omega^{-1}a'^{-1}_I \in \mathrm{Im}(\alpha)$, $a_I\omega_1 b'_I b'^{-1}_I\omega_1^{-1}\omega^{-1}a'^{-1}_I \notin \mathrm{Im}(\alpha)$. Now $(a_I\omega^{-1}a'^{-1}_I)^{-1}a_I\omega^{-1}a_I^{-1}(a_I\omega^{-1}a'^{-1}_I) = a'_I\omega^{-1}a'^{-1}_I \in \mathrm{Im}(\alpha)$, whereas $a_I\omega^{-1}a'^{-1}_I \notin \mathrm{Im}(\alpha)$ and $a_I\omega^{-1}a_I^{-1} \in \mathrm{Im}(\alpha)$. We thus get a contradiction to the malnormality of $\mathrm{Im}(\alpha)$ in F_n. This completes the proof. □

13.3 Proof of Theorem 13.2

From Lemmas 13.6 and 13.7, the Cayley complex $\mathcal{C}(G_\alpha)$ is the mapping-telescope of a strongly hyperbolic forest-map, equipped with the standard metric. A Cayley complex is connected. Thus, from Theorem 12.4, $\mathcal{C}(G_\alpha)$ is a Gromov-hyperbolic metric space for any mapping-telescope standard metric. From Lemma 13.5 the group G_α acts cocompactly, properly discontinuously and isometrically on $\mathcal{C}(G_\alpha)$ equipped with a mapping-telescope standard metric. A classical lemma of geometric group theory (usually attributed to Effremovich, Svàrc, Milnor – see [19] or [17] for instance), applied to quasi geodesic metric spaces, tells us that G_α and $\mathcal{C}(G_\alpha)$ are quasi-isometric so that G_α is a hyperbolic group. □

Remark 13.8. Another way of stating our main theorem about 'forest-stacks', using the language of trees of spaces, goes roughly as follows: "An oriented **R**-tree of **R**-trees with the gluing-maps satisfying the conditions of hyperbolicity and strong hyperbolicity with uniform constants is Gromov-hyperbolic." Here 'oriented **R**-tree' means an **R**-tree T equipped with an orientation going from the domain to the image of each attaching-map, and a surjective continuous map $f\colon T \to \mathbf{R}$ respecting this orientation. As a corollary of our theorem, and in order to illustrate it, we chose to concentrate on mapping-telescopes. We could as well consider spaces similar to mapping-telescopes but where we allow the attaching-maps not to be the same at each step. Our only requirement is to have uniform constants of quasi-isometry, hyperbolicity and so on. Also, with respect to groups, a corollary could have been stated dealing with HNN-extensions rather than just semi-direct products.

Another result which easily follows from our work could be more or less stated as follows. "Let T be a tree of spaces X_i, $i = 0, 1, \ldots$. Let $\psi\colon T \to T$ be a map of T such that the mapping-telescope of each X_i under ψ is Gromov-hyperbolic. If ψ induces a hyperbolic map on the tree resulting of the collapsing of each X_i to a point, then the mapping-telescope of the tree of spaces T under ψ is Gromov-hyperbolic." We leave the precise statement of such corollaries to the reader. Together with [14] where a new proof of the

Bestvina-Feighn theorem is given for mapping-tori of surface groups, the last one gives, thanks to [26], a new proof of the full version of the Combination Theorem for mapping-tori of hyperbolic groups, namely: "If G is a hyperbolic group and α is a hyperbolic automorphism of G, then $G \rtimes_\alpha \mathbf{Z}$ is a hyperbolic group."

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