

2.1 Chatterjee-Hitchin gerbes

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

The L_{ij} , together with these isomorphisms, define a gerbe over $SU(d+1)$, representing the generator of $H^3(SU(d+1), \mathbf{Z})$.

More generally, consider any compact, simply connected, simple Lie group G of rank d . Up to conjugacy, G contains exactly $d+1$ elements with semi-simple centralizer. (For $G = SU(d+1)$, these are the central elements.) Let $\mathcal{C}_1, \dots, \mathcal{C}_{d+1} \subset G$ be their conjugacy classes. We will define an invariant open cover V_1, \dots, V_{d+1} of G , with the property that each member of this cover admits an equivariant retraction onto the conjugacy class $\mathcal{C}_j \subset V_j$. It turns out that every semi-simple centralizer has a distinguished central extension by $U(1)$. This central extension defines an equivariant bundle gerbe on \mathcal{C}_j , hence (by pull-back) an equivariant bundle gerbe over V_j . We will find that these gerbes over V_j glue together to produce a gerbe over G , using a gluing rule developed in this paper.

The organization of the paper is as follows. In Section 2 we review the theory of gerbes and pseudo-line bundles with connections, and discuss 'strong equivariance' under a group action. Section 4 describes gluing rules for bundle gerbes. Section 3 summarizes some facts about gerbes coming from central extensions. In Section 5 we give the construction of the basic gerbe over G outlined above, and in Section 6 we study the 'pre-quantization of conjugacy classes'.

ACKNOWLEDGEMENTS. I would like to thank Ping Xu for fruitful discussions at the Poisson 2002 meeting in Lisbon, and for a preliminary version of his preprint [2] with Behrend and Zhang, giving yet another construction of the basic gerbe over G . Their (infinite-dimensional) approach is based on the notion of Morita equivalence of (quasi-)symplectic groupoids. I thank the referees for detailed comments and suggestions.

2. GERBES WITH CONNECTIONS

In this section we review gerbes on manifolds, along the lines of Chatterjee-Hitchin and Murray.

2.1 CHATTERJEE-HITCHIN GERBES

Let M be a manifold. Any Hermitian line bundle over M can be described by an open cover U_a , and transition functions $\chi_{ab}: U_a \cap U_b \rightarrow U(1)$ satisfying a cocycle condition $(\delta\chi)_{abc} = \chi_{bc}\chi_{ac}^{-1}\chi_{ab} = 1$ on triple intersections. The

cohomology class in $H^1(M, \underline{U(1)}) = H^2(M, \mathbf{Z})$ defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in $H^3(M, \mathbf{Z})$ in a similar fashion, replacing $U(1)$ -valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles $L_{ab} \rightarrow U_a \cap U_b$ and a trivialization, i.e. unit length section, t_{abc} of the line bundle $(\delta L)_{abc} = L_{bc}L_{ac}^{-1}L_{ab}$ over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd}t_{acd}^{-1}t_{abd}t_{abc}^{-1} = 1,$$

which makes sense since $(\delta t)_{abcd}$ is a section of the *canonically* trivial bundle. (Each factor L_{ab} cancels with a factor L_{ab}^{-1} .) After passing to a refinement of the cover, such that all L_{ab} become trivializable, and picking trivializations, t_{abc} is simply a Čech cocycle of degree 2, hence defines a class in $H^2(M, \underline{U(1)}) = H^3(M, \mathbf{Z})$. The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if M is a connected, oriented 3-manifold, the generator of $H^3(M, \mathbf{Z}) = \mathbf{Z}$ can be described in terms of the cover U_1, U_2 , where U_1 is an open ball around a given point $p \in M$, and $U_2 = M \setminus \{p\}$, using the degree one line bundle over $U_1 \cap U_2 \cong S^2 \times (0, 1)$.

2.2 BUNDLE GERBES

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let $\pi: X \rightarrow M$ be a fiber bundle, or more generally a surjective submersion. (Different components of X may have different dimensions.) For each $k \geq 0$ let $X^{[k]}$ denote the k -fold fiber product of X with itself. There are $k + 1$ projections $\partial^i: X^{[k+1]} \rightarrow X^{[k]}$, omitting the i th factor in the fiber product. Suppose we are given a smooth function $\chi: X^{[2]} \rightarrow U(1)$, satisfying a cocycle condition $\delta\chi = 1$ where

$$\delta\chi := \partial_0^*\chi\partial_1^*\chi^{-1}\partial_2^*\chi: X^{[3]} \rightarrow U(1).$$

Then χ determines a Hermitian line bundle $L \rightarrow M$, with fibers at $m \in M$ the space of all linear maps $\phi: X_m = \pi^{-1}(m) \rightarrow \mathbf{C}$ such that $\phi(x) = \chi(x, x')\phi(x')$. Given local sections $\sigma_a: U_a \rightarrow X$ of X , the pull-backs of χ under the maps $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$ give transition functions χ_{ab} for the line bundle.

Again, replacing $U(1)$ -valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle $L \rightarrow X^{[2]}$ and a trivializing section t of the line bundle $\delta L = \partial_0^*L \otimes \partial_1^*L^{-1} \otimes \partial_2^*L$