

## 2.2 Bundle gerbes

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cohomology class in  $H^1(M, \underline{U(1)}) = H^2(M, \mathbf{Z})$  defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in  $H^3(M, \mathbf{Z})$  in a similar fashion, replacing  $U(1)$ -valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles  $L_{ab} \rightarrow U_a \cap U_b$  and a trivialization, i.e. unit length section,  $t_{abc}$  of the line bundle  $(\delta L)_{abc} = L_{bc}L_{ac}^{-1}L_{ab}$  over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd}t_{acd}^{-1}t_{abd}t_{abc}^{-1} = 1,$$

which makes sense since  $(\delta t)_{abcd}$  is a section of the *canonically* trivial bundle. (Each factor  $L_{ab}$  cancels with a factor  $L_{ab}^{-1}$ .) After passing to a refinement of the cover, such that all  $L_{ab}$  become trivializable, and picking trivializations,  $t_{abc}$  is simply a Čech cocycle of degree 2, hence defines a class in  $H^2(M, \underline{U(1)}) = H^3(M, \mathbf{Z})$ . The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if  $M$  is a connected, oriented 3-manifold, the generator of  $H^3(M, \mathbf{Z}) = \mathbf{Z}$  can be described in terms of the cover  $U_1, U_2$ , where  $U_1$  is an open ball around a given point  $p \in M$ , and  $U_2 = M \setminus \{p\}$ , using the degree one line bundle over  $U_1 \cap U_2 \cong S^2 \times (0, 1)$ .

## 2.2 BUNDLE GERBES

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let  $\pi: X \rightarrow M$  be a fiber bundle, or more generally a surjective submersion. (Different components of  $X$  may have different dimensions.) For each  $k \geq 0$  let  $X^{[k]}$  denote the  $k$ -fold fiber product of  $X$  with itself. There are  $k + 1$  projections  $\partial^i: X^{[k+1]} \rightarrow X^{[k]}$ , omitting the  $i$ th factor in the fiber product. Suppose we are given a smooth function  $\chi: X^{[2]} \rightarrow U(1)$ , satisfying a cocycle condition  $\delta\chi = 1$  where

$$\delta\chi := \partial_0^*\chi\partial_1^*\chi^{-1}\partial_2^*\chi: X^{[3]} \rightarrow U(1).$$

Then  $\chi$  determines a Hermitian line bundle  $L \rightarrow M$ , with fibers at  $m \in M$  the space of all linear maps  $\phi: X_m = \pi^{-1}(m) \rightarrow \mathbf{C}$  such that  $\phi(x) = \chi(x, x')\phi(x')$ . Given local sections  $\sigma_a: U_a \rightarrow X$  of  $X$ , the pull-backs of  $\chi$  under the maps  $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$  give transition functions  $\chi_{ab}$  for the line bundle.

Again, replacing  $U(1)$ -valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle  $L \rightarrow X^{[2]}$  and a trivializing section  $t$  of the line bundle  $\delta L = \partial_0^*L \otimes \partial_1^*L^{-1} \otimes \partial_2^*L$

over  $X^{[3]}$ , satisfying a compatibility condition  $\delta t = 1$  over  $X^{[4]}$  (which makes sense since  $\delta t$  is a section of the canonically trivial bundle  $\delta\delta L$ ). Given local sections  $\sigma_a: U_a \rightarrow X$ , one can pull these data back under the maps  $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$  and  $(\sigma_a, \sigma_b, \sigma_c): U_a \cap U_b \cap U_c \rightarrow X^{[3]}$  to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of  $(X, L, t)$  is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with  $X$  the disjoint union of the sets  $U_a$  in the given cover.

REMARK 2.1. In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called ‘locally split’) is used that every point  $x \in M$  admits an open neighborhood  $U$  and a map  $\sigma: U \rightarrow X$  such that  $\pi \circ \sigma = \text{id}$ . However, this condition seems insufficient in the smooth category, as the fiber product  $X \times_M X$  need not be a manifold unless  $\pi$  is a submersion.

### 2.3 SIMPLICIAL GERBES

Murray’s construction fits naturally into a wider context of *simplicial gerbes*. We refer to Mostow-Perchik’s notes of lectures by R. Bott [23] and to Dupont’s paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold*  $M_\bullet$  is a sequence of manifolds  $(M_n)_{n=0}^\infty$ , together with *face maps*  $\partial_i: M_n \rightarrow M_{n-1}$  for  $i = 0, \dots, n$  satisfying relations  $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$  for  $i < j$ . (The standard definition also involves *degeneracy maps* but these need not concern us here.) The *(fat) geometric realization* of  $M_\bullet$  is the topological space  $\|M\| = \coprod_{n=1}^\infty \Delta^n \times M_n / \sim$ , where  $\Delta^n$  is the  $n$ -simplex and the relation is  $(t, \partial_i(x)) \sim (\partial^i(t), x)$ , for  $\partial^i: \Delta^{n-1} \rightarrow \Delta^n$  the inclusion as the  $i$ th face. A (smooth) simplicial map between simplicial manifolds  $M_\bullet, M'_\bullet$  is a collection of smooth maps  $f_n: M_n \rightarrow M'_n$  intertwining the face maps; such a map induces a map between the geometric realizations.

#### EXAMPLES 2.2.

(a) If  $S$  is any manifold, one can define a simplicial manifold  $E_\bullet S$  where  $E_n S$  is the  $n + 1$ -fold cartesian product of  $S$ , and  $\partial_j$  omits the  $j$ th factor. It is known [23] that the geometric realization  $\|ES\|$  of this simplicial manifold is contractible. More generally, if  $X \rightarrow M$  is a fiber bundle with fiber  $S$ ,