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Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the δ-cohomology of the double complex $\Omega^k(M_n)$ vanishes in bidegrees $(1,2)$ and $(2,1)$. In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion $\pi\colon X \to M$ the sequence

$$(2.1) \qquad 0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \cdots$$

is exact, so the δ-cohomology vanishes in *all* degrees.

Thus, every bundle gerbe $\mathcal{G} = (X,L,t)$ over a manifold M (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the 3-*curvature* $\eta \in \Omega^3(M)$ of the bundle gerbe connection by $\pi^*\eta = \mathrm{d}B \in \ker\delta$. It can be shown that its cohomology class is the image of the Dixmier-Douady class $[\mathcal{G}]$ under the map $H^3(M,\mathbf{Z}) \to H^3(M,\mathbf{R})$. Similarly, if $\mathcal{G}$ admits a pseudo-line bundle $\mathcal{L} = (E,s)$, one can always choose a pseudo-line bundle connection ∇^E. The difference $\frac{1}{2\pi i}\operatorname{curv}(\nabla^E) - B$ is δ-closed and one defines the *error* 2-*form* of this connection by

$$\pi^*\omega = \frac{1}{2\pi i}\operatorname{curv}(\nabla^E) - B\,.$$

It is clear from the definition that $\mathrm{d}\omega + \eta = 0$.

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

2.4 EQUIVARIANT BUNDLE GERBES

Suppose G is a Lie group acting on X and on M, and that $\pi\colon X \to M$ is a G-equivariant surjective submersion. Then G acts on all fiber products $X^{[p]}$. We will say that a bundle gerbe $\mathcal{G} = (X,L,t)$ is G-*equivariant*, if L is a G-equivariant line bundle and t is a G-invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction[1]) $X_G = EG \times_G X \to M_G = EG \times_G M$, hence has an *equivariant* Dixmier-Douady class in $H^3(M_G,\mathbf{Z}) = H^3_G(M,\mathbf{Z})$. Similarly, we say that a pseudo-line bundle (E,s) for (X,L,t) is equivariant, provided E carries a G-action and s is an invariant section.

[1]) We have not discussed bundle gerbes over infinite-dimensional spaces such as M_G. Recall however [4] that the classifying bundle $EG \to BG$ may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.

REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if $X = \coprod U_a$, for an open cover $\mathcal{U} = \{U_a, a \in A\}$, a G-action on X would amount to the cover being G-invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group G, the equivariant cohomology $H_G^\bullet(M, \mathbf{R})$ may be computed from Cartan's complex of equivariant differential forms $\Omega_G^\bullet(M)$, consisting of G-equivariant polynomial maps $\alpha\colon \mathfrak{g} \to \Omega(M)$. The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(\mathrm{d}_G\, \alpha)(\xi) = \mathrm{d}\, \alpha(\xi) - \iota(\xi_M)\alpha(\xi)\,,$$

where $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)$ is the generating vector field corresponding to $\xi \in \mathfrak{g}$. Given a G-equivariant connection ∇^L on an equivariant line bundle, one defines [3, Chapter 7] a d_G-closed equivariant curvature $\mathrm{curv}_G(\nabla^L) \in \Omega_G^2(M)$.

A equivariant connection on a G-equivariant bundle gerbe (X, L, t) over M is a pair (∇^L, B_G), where ∇^L is an invariant connection and $B_G \in \Omega_G^2(X)$ an equivariant 2-form, such that $\delta\nabla^L t = 0$ and $\delta B_G = \frac{1}{2\pi i}\mathrm{curv}_G(\nabla^L)$. Its equivariant 3-curvature $\eta_G \in \Omega_G^3(M)$ is defined by $\pi^*\eta_G = \mathrm{d}_G B_G$. Given an *invariant* pseudo-line bundle connection ∇^E on a equivariant pseudo-line bundle (E, s), one defines the equivariant error 2-form ω_G by

$$\pi^*\omega_G = \frac{1}{2\pi i}\mathrm{curv}_G(\nabla^E) - B_G\,.$$

Clearly, $\mathrm{d}_G\,\omega_G + \eta_G = 0$.

3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over G. Suppose $\mathrm{U}(1) \to \widehat{K} \to K$ is a central extension, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. Given a principal K-bundle $\pi\colon P \to B$, one constructs a bundle gerbe (P, L, t), sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K\,,$$