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5.2 THE BASIC 3-FORM ON  $G$ 

Let  $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$  be the left- and right-invariant Maurer-Cartan forms on  $G$ , respectively. The 3-form  $\eta \in \Omega^3(G)$  given by<sup>3)</sup>

$$\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12} \theta^R \cdot [\theta^R, \theta^R]$$

is closed, and has a closed equivariant extension  $\eta_G \in \Omega_G^3(G)$  given by

$$\eta_G(\xi) := \eta - \frac{1}{2}(\theta^L + \theta^R) \cdot \xi.$$

Their cohomology classes represent generators of  $H^3(G, \mathbf{Z}) = \mathbf{Z}$  and  $H_G^3(G, \mathbf{Z}) = \mathbf{Z}$ , respectively. The pull-back of  $\eta_G$  to any conjugacy class  $\iota_C: C \hookrightarrow G$  is exact. In fact, let  $\omega_C \in \Omega^2(C)^G \subset \Omega_G^2(C)$  be the invariant 2-form given on generating vector fields  $\xi_C, \xi'_C$  for  $\xi, \xi' \in \mathfrak{g}$  by the formula

$$\omega_C(\xi_C(g), \xi'_C(g)) = \frac{1}{2} \xi \cdot (\text{Ad}_g - \text{Ad}_{g^{-1}}) \xi'.$$

Then [1, 16]

$$d_G \omega_C + \iota_C^* \eta_G = 0.$$

We will now show that  $\eta_G$  is exact over each of the open subsets  $V_j$ . Let  $C_j = q^{-1}(\mu_j) \subset V_j$  be the conjugacy classes corresponding to the vertices.

LEMMA 5.1. *The linear retraction*

$$[0, 1] \times \mathfrak{A}_j \rightarrow \mathfrak{A}_j, \quad (t, \mu_j + \zeta) \mapsto \mu_j + (1 - t) \zeta$$

of  $\mathfrak{A}_j$  onto the vertex  $\mu_j$  lifts uniquely to a smooth  $G$ -equivariant retraction from  $V_j$  onto  $C_j$ .

*Proof.* Recall that the slice  $S_j$  is an open neighborhood of  $\exp(\mu_j)$  in  $G_j$ . Any  $G_j$ -equivariant retraction from  $S_j$  onto  $\exp \mu_j$  extends uniquely to a  $G$ -equivariant retraction from  $V_j = G \times_{G_j} S_j$  onto  $C_j$ . Note that  $S'_j = G_j \cdot (\mathfrak{A}_j - \mu_j)$  is a star-shaped open neighborhood of 0 in  $\mathfrak{g}_j$ , and that  $S'_j \rightarrow S_j, \zeta \mapsto \exp(\mu_j) \exp(\zeta)$  is a  $G_j$ -equivariant diffeomorphism. The linear retraction of  $S'_j$  onto the origin gives the desired retraction of  $S_j$ . Uniqueness is clear, since the retraction has to preserve  $\exp(\mathfrak{A}_j) \subset V_j$ , by equivariance.

<sup>3)</sup> For  $\mathfrak{g}$ -valued forms  $\beta_1, \beta_2$ , the bracket  $[\beta_1, \beta_2]$  denotes the  $\mathfrak{g}$ -valued form obtained by applying the Lie bracket  $[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  to the  $\mathfrak{g} \otimes \mathfrak{g}$ -valued form  $\beta_1 \wedge \beta_2$ .

Let

$$\mathbf{h}_j: \Omega^p(V_j) \rightarrow \Omega^p([0, 1] \times V_j) \rightarrow \Omega^{p-1}(V_j)$$

be the de Rham homotopy operator for this retraction, given (up to a sign) by pull-back under the retraction, followed by integration over the fibers of  $[0, 1] \times V_j \rightarrow V_j$ . It has the property

$$(5.1) \quad d_G \mathbf{h}_j + \mathbf{h}_j d_G = \text{Id} - \pi_j^* \iota_j^*$$

where  $\iota_j: C_j \rightarrow V_j$  is the inclusion and  $\pi_j: V_j = G \times_{G_j} S_j \rightarrow G/G_j = C_j$  the projection. Let  $(\varpi_j)_G = \mathbf{h}_j \eta_G - \pi_j^* \omega_{C_j} \in \Omega_G^2(V_j)$ , and write  $(\varpi_j)_G = \varpi_j - \Psi_j$  where  $\varpi_j \in \Omega^2(V_j)$  and  $\Psi_j \in \Omega^0(V_j, \mathfrak{g})$ .

**PROPOSITION 5.2.** *The equivariant 2-form  $(\varpi_j)_G = \varpi_j - \Psi_j$  has the following properties.*

(a)  $d_G(\varpi_j)_G = \eta_G$ .

(b) *The pull-back of  $(\varpi_j)_G$  to a conjugacy class  $C \subset V_j$  is given by*

$$\iota_C^*(\varpi_j)_G = \Psi_j^*(\omega_{\mathcal{O}})_G - \omega_C,$$

where  $(\omega_{\mathcal{O}})_G$  is the equivariant symplectic form on the adjoint orbit  $\mathcal{O} = \Psi_j(C)$ ,

(c) *The pull-back of  $\Psi_j$  to the conjugacy class  $C_j$  vanishes. In fact,  $\Psi_j(\exp \xi) = \xi - \mu_j$  for all  $\xi \in \mathfrak{A}_j$ .*

(d) *Over each intersection  $V_{ij} = V_i \cap V_j$ , the difference  $\Psi_i - \Psi_j$  takes values in the adjoint orbit  $\mathcal{O}_{ij}$  through  $\mu_j - \mu_i \in \mathfrak{g} \cong \mathfrak{g}^*$ . Furthermore,*

$$(\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G$$

where  $p_{ij}: V_{ij} \rightarrow \mathcal{O}_{ij}$  is the map defined by  $\Psi_i - \Psi_j$ , and  $(\omega_{\mathcal{O}_{ij}})_G$  is the equivariant symplectic form on the orbit.

*Proof.* (a) holds by construction. (b) follows from the observation that  $\iota_C^*(\varpi_j)_G + \omega_C$  is an equivariantly closed 2-form on  $C_j$ , with  $\Psi_j$  as its moment map. To prove (c) we note that since the retraction is equivariant, we have  $\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* = (\exp|_{\mathfrak{A}_j})^* \circ \mathbf{h}_j$  where  $(\exp|_{\mathfrak{A}_j})^*$  is pull-back to  $\mathfrak{A}_j \subset \mathfrak{t}$  and where  $\tilde{\mathbf{h}}_j$  is the homotopy operator for the linear retraction of  $\mathfrak{t}$  onto  $\{\mu_j\}$ . Let  $\nu: \mathfrak{A}_j \rightarrow \mathfrak{t}$  be the coordinate function (inclusion). Then

$$\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* \frac{1}{2}(\theta^L + \theta^R) = \tilde{\mathbf{h}}_j \circ d\nu = \nu - \mu_j,$$

proving that  $(\exp|_{\mathfrak{A}_j})^* \Psi_j = \nu - \mu_j$ . This yields (c), by equivariance. For  $\nu \in \mathfrak{A}_{ij}$  we have, using (c),

$$(\Psi_i - \Psi_j)(\exp \nu) = (\nu - \mu_i) - (\nu - \mu_j) = \mu_j - \mu_i.$$

By equivariance, it follows that  $\Psi_i - \Psi_j$  takes values in the adjoint orbit through  $\mu_j - \mu_i$ . The difference  $\varpi_i - \varpi_j$  vanishes on the maximal torus  $T$ , and is therefore determined by its contractions with generating vector fields. Since  $\Psi_i - \Psi_j$  is a moment map for  $\varpi_i - \varpi_j$ , it follows that  $\varpi_i - \varpi_j$  equals the pull-back of the symplectic form on  $G \cdot (\mu_j - \mu_i)$ .

### 5.3 THE SPECIAL UNITARY GROUP

For the special unitary group  $G = \mathrm{SU}(d+1)$ , the construction of the basic gerbe simplifies due to the fact that in this case all vertices  $\mu_j$  of the alcove are contained in the weight lattice. In fact the gerbe is presented as a Chatterjee-Hitchin gerbe for the cover  $\mathcal{V} = \{V_i, i = 0, \dots, d\}$ .

For each weight  $\mu \in \Lambda^* \subset \mathfrak{t} \subset \mathfrak{g}$ , let  $G_\mu$  be its stabilizer for the adjoint action and let  $\mathbf{C}_\mu$  the 1-dimensional  $G_\mu$ -representation with infinitesimal character  $\mu$ . Let the line bundle  $L_\mu = G \times_{G_\mu} \mathbf{C}_\mu$  equipped with the unique left-invariant connection  $\nabla$ . Then  $L_\mu$  is a  $G$ -equivariant pre-quantum line bundle for the orbit  $\mathcal{O} = G \cdot \mu$ . That is,

$$\frac{i}{2\pi} \mathrm{curv}_G(\nabla) = (\omega_{\mathcal{O}})_G := \omega_{\mathcal{O}} - \Phi_{\mathcal{O}}$$

where  $\omega_{\mathcal{O}}$  is the symplectic form and  $\Phi_{\mathcal{O}}: \mathcal{O} \hookrightarrow \mathfrak{g}^*$  is the moment map given as inclusion.

In particular, in the case of  $\mathrm{SU}(d+1)$  all orbits  $\mathcal{O}_{ij} = G \cdot (\mu_j - \mu_i)$  carry  $G$ -equivariant pre-quantum line bundles. Recall the fibrations  $p_{ij}: V_{ij} \rightarrow \mathcal{O}_{ij}$  defined by  $\Psi_i - \Psi_j$ , and let

$$L_{ij} = p_{ij}^*(L_{\mu_j - \mu_i}),$$

equipped with the pull-back connection. For any triple intersection  $V_{ijk} = G \times_{G_{ijk}} S_{ijk}$ , the tensor product  $(\delta L)_{ijk} = L_{jk} L_{ik}^{-1} L_{ij}$  is the pull-back of the line bundle over  $G/G_{ijk}$ , defined by the zero weight

$$(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0$$

of  $G_{ijk}$ . It is hence canonically trivial, with  $(\delta \nabla)_{ijk}$  the trivial connection. The trivializing section  $t_{ijk} = 1$  satisfies  $\delta t = 1$  and  $(\delta \nabla)t = 0$ . Take  $(B_j)_G = (\varpi_j)_G$ . Then

$$(B_j)_G - (B_i)_G = (\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G = \frac{1}{2\pi i} \mathrm{curv}_G(\nabla^{L_{ij}}).$$

Thus  $\mathcal{G} = (\mathcal{V}, L, t)$  is a equivariant gerbe with connection  $(\nabla, B)$ . Since