6. Pre-quantization of conjugacy classes

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a distinguished, equivariant pseudo-line bundle (E_{ij}, s_{ij}) (where E_{ij} is trivial), with connection $\nabla^{E_{ij}}$ induced from the connection θ_{ij}. From the definition of θ_{ij}, it follows that the equivariant error 2-form for this connection is the pull-back of the equivariant symplectic form on the coadjoint orbit through $\mu_j - \mu_i$.

We now modify the bundle gerbe connection by adding the equivariant 2-form $(\varpi_j)_G \in \Omega^2_G(V_j)$ to the gerbe connection. Proposition 5.2(d) shows that the equivariant error 2-form of $\nabla^{E_{ij}}$ with respect to the new gerbe connection vanishes. The other conditions from the gluing construction in §4 are trivially satisfied. Since the equivariant 3-curvature for the new gerbe connection on $\mathcal{G}_j$ is $d_G(\varpi_j)_G = \eta_G|_{V_j}$, we have constructed an equivariant bundle gerbe with connection, with equivariant curvature-form η_G.

REMARK 5.6. For $G = \mathrm{SU}(d+1)$ this construction reduces to the construction in terms of transition line bundles: All L_i, t_i, E_{ij}, u_{ijk} are trivial in this case, hence the entire information on the gerbe resides in the functions $s_{ij}\colon (X_{ij})^{[2]} \to \mathrm{U}(1)$ defined by the differences $\mu_j - \mu_i$. The condition $\delta s_{ij} = 1$ for these functions means that s_{ij} defines a line bundle L_{ij} over V_{ij}, as remarked at the beginning of Section 2.2. The condition $s_{ij}s_{jk}s_{ki} = 1$ over X_{ijk} is the compatibility condition over triple intersections.

6. PRE-QUANTIZATION OF CONJUGACY CLASSES

It is a well-known fact from symplectic geometry that a coadjoint orbit $\mathcal{O} = G.\mu$ through $\mu \in \mathfrak{t}^*_+$ has integral symplectic form, i.e. admits a pre-quantum line bundle, if and only if μ is in the weight lattice Λ^*. The analogous question for conjugacy classes reads: For which $\mu \in \mathfrak{A}$ and $m \in \mathbf{N}$ does the pull-back of the mth power of the basic gerbe $\mathcal{G}^m$ to the conjugacy class $\mathcal{C} = G.\exp(\mu)$ admit a pseudo-line bundle, with $m\omega_{\mathcal{C}}$ as its error 2-form? For any positive integer $m > 0$ let

$$\Lambda^*_m = \Lambda^* \cap m\mathfrak{A}$$

be the set of level m weights. As is well-known [26], the set Λ^*_m parametrizes the positive energy representations of the loop group LG at level m.

THEOREM 6.1. *The restriction of $\mathcal{G}^m$ to a conjugacy class $\mathcal{C}$ admits a pseudo-line bundle $\mathcal{L}$ with connection, with error 2-form $m\omega_{\mathcal{C}}$, if and only if $\mathcal{C} = G.\exp(\mu/m)$ with $\mu \in \Lambda^*_m$. Moreover $\mathcal{L}$ has an equivariant extension in this case, with $m\omega_{\mathcal{C}}$ as its* equivariant *error 2-form .*

Proof. Given a conjugacy class $\mathcal{C} \subset G$, let $\mu \in m\mathfrak{A}$ be the unique point with $g := \exp(\mu/m) \in \mathcal{C}$, and let $K = G_g$ so that $\mathcal{C} = G/K$. Pick an index j with $\mathcal{C} \subset V_j$, and let

$$\nu = m\Psi_j(g) = \mu - m\mu_j .$$

Then

$$G_\mu \subset K \subset G_\nu .$$

Let $\mathcal{O}_\mu, \mathcal{O}_\nu \subset \mathfrak{g}$ denote the adjoint orbits of μ, ν, and $(\omega_\mu)_G$, $(\omega_\nu)_G$ their equivariant symplectic forms. The pull-back $\iota_{\mathcal{C}}^*\mathcal{G}^m$ is the gerbe over G/K defined as in Section 3 by the homomorphism $\varrho \in \operatorname{Hom}(\pi_1(K), \mathrm{U}(1))$, given as a composition

$$\pi_1(K) \to \pi_1(G_j) \to \mathrm{U}(1) ,$$

where the fist map is push-forward under the inclusion $K \hookrightarrow G_j$, and the second map is the homomorphism defined by the element $m\mu_j \in \mathfrak{t}$ for G_j.

Suppose now that $\mu \in \Lambda_m^*$. Then $m\mu_j$ equals $-\nu$ up to a weight lattice vector, which means that ϱ is the image of $-\nu \in (\mathfrak{k}^*)^K$ in the exact sequence (3.2). Hence, Proposition 3.2 says that we we obtain an equivariant pseudo-line bundle for $\iota_{\mathcal{C}}^*\mathcal{G}^m$, with equivariant error 2-form

$$\Psi_j^*(\omega_\nu)_G - m\, \iota_{\mathcal{C}}^*(\varpi_j)_G = m\,\omega_{\mathcal{C}} .$$

Here we have used part (b) of Proposition 5.2.

Conversely, suppose that $\mathcal{G}^m|_{\mathcal{C}}$ admits a pseudo-line bundle with error 2-form $m\,\omega_{\mathcal{C}}$. Consider the pull-back of $\mathcal{G}$ under the exponential map $\exp\colon \mathfrak{g} \to G$. The pull-back $\exp^*\eta \in \Omega^3(\mathfrak{g})$ is exact, and the homotopy operator for the linear retraction of $\mathfrak{g}$ to the origin defines a 2-form $\varpi \in \Omega^2(\mathfrak{g})$ with $\mathrm{d}\varpi = \exp^*\eta$. As in Proposition 5.2, one shows that for any adjoint orbit $\mathcal{O} \subset \mathfrak{g}$, with $\exp\mathcal{O} = \mathcal{C}$,

$$\iota_{\mathcal{O}}^*\varpi = \exp^*\omega_{\mathcal{C}} - \omega_{\mathcal{O}}$$

where $\omega_{\mathcal{O}}$ is the symplectic form on $\mathcal{O}$. In particular this applies to $\mathcal{O} = \mathcal{O}_{\mu/m}$. Choose a pseudo-line bundle for $\exp^*\mathcal{G}$ with error 2-form $-\varpi$. We then have two pseudo-line bundles for $\exp^*\mathcal{G}^m|_{\mathcal{O}}$ obtained by restricting the mth power of the pseudo-line bundle for $\exp^*\mathcal{G}$ or by pulling back the pseudo-line bundle for $\mathcal{C}$. Their quotient is a line bundle over $\mathcal{O}$, with curvature the difference of the error 2-forms:

$$m(\exp^*\omega_{\mathcal{C}} - \iota_{\mathcal{O}_\mu}^*\varpi) = m\omega_{\mathcal{O}} .$$

Thus $m(\mu/m) = \mu$ must be in the weight lattice.

REMARK 6.2. Z. Shahbazi has proved that if $\mathcal{G}$ is a gerbe with connection over a manifold M, with curvature 3-form η, and $\Phi\colon N \to M$ is a map with $\Phi^*\eta + d\omega = 0$, then the pull-back gerbe $\Phi^*\mathcal{G}$ admits a pseudo-line bundle, with ω as its error 2-form, if and only if the pair (η, ω) defines an integral element of the relative de Rham cohomology $H^3(\Phi, \mathbf{R})$. This means that for any smooth 2-cycle $S \subset N$, and any smooth 3-chain $B \subset M$ with boundary $\Phi(S)$, one must have $\int_B \eta - \int_S \omega \in \mathbf{Z}$. The particular case where the target of Φ is a Lie group G is relevant for the pre-quantization of group-valued moment maps [1].

APPENDIX A. PROOF OF LEMMA 4.4

In this Appendix we prove Lemma 4.4, concerning the construction of a certain cover U_I of M from a given cover V_j. Write $M = \coprod_I A_I$ where

$$A_I = \bigcap_{i\in I} V_i \backslash \bigcup_{j\notin I} V_j .$$

Notice that $\bar{A}_I \subset \bigcup_{J\subset I} A_J$. By induction on the cardinality $k = |I|$ we will construct open sets $U_J \subset V_J$, having the following properties:

(a) the closure $\bar{U}_I$ does not meet $\bar{U}_J$ for $|J| \le |I|$ unless $J \subset I$,

(b) each $\bar{A}_I$ is contained in the union of U_J with $J \subset I$.

The induction starts at $k = 0$, taking $U_\varnothing = \varnothing$. Suppose we have constructed open sets U_I with $\bar{U}_I \subset V_I$ for $|I| < k$, such that the properties (a), (b) hold for all $|I| < k$. For $|I| = k$ consider the subsets

$$B_I := A_I \backslash \Big(\bigcup_{J\subset I, |J|<k} U_J \Big) .$$

Note that (unlike A_I) the set B_I is closed. B_I does not meet $\bar{A}_J$ unless $I \subset J$, and it also does not meet $\bar{U}_J$ for $|J| < k$ unless $J \subset I$. That is, B_I is disjoint from

$$C_I := \bigcup_{J\not\subset I, |J|<k} \bar{U}_J \cup \bigcup_{K\not\supset I} \bar{A}_K .$$

Choose open sets U_I for $|I| = k$ with $B_I \subset U_I \subset \bar{U}_I \subset M\backslash C_I$, and such that the closures of the sets U_I for distinct I with $|I| = k$ are disjoint. The new collection of subsets will satisfy the properties (a), (b) for $|I| \le k$. We next show that $V_i' = M\backslash \bigcup_{J\not\ni i} \bar{U}_J$ is a cover of M. Write $M = \coprod_I D_I$ with $D_I = \bar{U}_I \backslash \bigcup_{|J|<|I|} \bar{U}_J$. Then $D_I \cap \bar{U}_J = \varnothing$ unless $I \subset J$, so D_I is contained