Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **08.08.2024**

L'Enseignement Mathématique, t. 49 (2003), p. 35–65

LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA

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INTRODUCTION

This paper arose from a series of three lectures given by the first author at the Università di Roma "Tor Vergata" in January 2002, when the second author extended and improved her notes of these lectures. It contains an elementary introduction for non-specialists to the theory of quasi-invariants (but no original results).

Our main object of study is the variety X_m of quasi-invariants for a finite Coxeter group. This very interesting singular algebraic variety arose in work of O. Chalykh and A. Veselov about 10 years ago, as the spectral variety of the quantum Calogero-Moser system. We will see that despite being singular, this variety has very nice properties (Cohen-Macaulay, Gorenstein, simplicity of the ring of differential operators, explicitly given Hilbert series). One should remark that although the definition of X_m is completely elementary, it is helpful, in order to understand the geometry of X_m, to use representation theory of the rational degeneration of Cherednik's double affine Hecke algebra, and the theory of integrable systems. Thus, the study of X_m leads us to a junction of three subjects — integrable systems, representation theory, and algebraic geometry. The content of the paper is as follows. In Lecture 1 we define the ring of quasi-invariants for a Coxeter group, and discuss its elementary properties (with proofs), as well as deeper properties, such as Cohen-Macaulay, the Gorenstein property, and the Hilbert series (whose partial

*) The work of the first author was partially supported by his NSF grant DMS-9988796 and was done in part for the Clay Mathematics Institute. The series of lectures in "Tor Vergata" was financed by the MIUR "Progetto Azioni di Gruppi" of which the second author is a member. The authors are also grateful to Corrado De Concini for useful discussions.

proofs are postponed until Lecture 3). In Lecture 2, we explain the origin of the ring of quasi-invariants in the theory of integrable systems, and introduce some tools from integrable systems, such as the Baker-Akhieser function. Finally, in Lecture 3, we develop the theory of the rational Cherednik algebra, the representation-theoretic techniques due to Opdam and Rouquier, and finish the proofs of the geometric statements from Chapter 1.

1. Lecture 1

1.1 Definition of quasi-invariants

In this lecture we will define the ring of quasi-invariants Q_m and discuss its main properties.

We will work over the field $\mathbf{C}$ of complex numbers. Let W be a finite Coxeter group, i.e. a finite group generated by reflections. Let us denote by $\mathfrak{h}$ its reflection representation. A typical example is the Weyl group of a semisimple Lie algebra acting on a Cartan subalgebra $\mathfrak{h}$. In the case the Lie algebra is $\mathfrak{sl}(n)$, we have that W is the symmetric group S_n on n letters and $\mathfrak{h}$ is the space of diagonal traceless $n \times n$ matrices.

Let $\Sigma \subset W$ denote the set of reflections. Clearly, W acts on Σ by conjugation. Let $m\colon \Sigma \to \mathbf{Z}_+$ be a function on Σ taking non negative integer values, which is W-invariant. The number of orbits of W on Σ is generally very small. For example, if W is the Weyl group of a simple Lie algebra of ADE type, then W acts transitively on Σ, so m is a constant function.

For each reflection $s \in \Sigma$, choose $\alpha_s \in \mathfrak{h}^* - \{0\}$ so that, for $x \in \mathfrak{h}$, $\alpha_s(sx) = -\alpha_s(x)$ (this means that the hyperplane given by the equation $\alpha_s = 0$ is the reflection hyperplane for s).

Definition 1.1 ([CV1, CV2]). A polynomial $q \in \mathbf{C}[\mathfrak{h}]$ is said to be *m-quasi-invariant* with respect to W if, for any $s \in \Sigma$, the polynomial $q(x) - q(sx)$ is divisible by $\alpha_s(x)^{2m_s+1}$.

We will denote by Q_m the space of m-quasi-invariant polynomials with respect to W.

Notice that every element of $\mathbf{C}[\mathfrak{h}]$ is a 0-quasi-invariant, and that every W-invariant is an m-quasi-invariant for any m. Indeed if $q \in \mathbf{C}[\mathfrak{h}]^W$, then we have $q(x) - q(sx) = 0$ for all $s \in \Sigma$, and 0 is divisible by all powers of $\alpha_s(x)$. Thus in a way, $\mathbf{C}[\mathfrak{h}]^W$ can be viewed as the set of ∞-quasi-invariants.