Zeitschrift: L'Enseignement Mathématique
Band: 49 (2003)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE CLASSIFICATION OF RATIONAL KNOTS
Kapitel: 1. Introduction
Autor: Kauffman, Louis H. / Lambropoulou, Sofia
DOI: https://doi.org/10.5169/seals-66693

L'Enseignement Mathématique, t. 49 (2003), p. 357–410

ON THE CLASSIFICATION OF RATIONAL KNOTS

by Louis H. KAUFFMAN and Sofia LAMBROPOULOU

ABSTRACT. In this paper we give combinatorial proofs of the classification of unoriented and oriented rational knots based on the now known classification of alternating knots and the calculus of continued fractions. We also characterize the class of strongly invertible rational links. Rational links are of fundamental importance in the study of DNA recombination.

1. INTRODUCTION

Rational knots and links comprise the simplest class of links. The first twenty five knots, except for 8_5, are rational. Furthermore all knots and links up to ten crossings are either rational or are obtained by inserting rational tangles into a small number of planar graphs, see [6]. Rational links are alternating with one or two unknotted components, and they are also known in the literature as Viergeflechte, four-plats or 2-bridge knots depending on their geometric representation. More precisely, rational knots can be represented as:

• plat closures of four-strand braids (Viergeflechte [1], four-plats). These are knot diagrams with two local maxima and two local minima.

• 2-bridge knots. A 2-bridge knot is a knot that has a diagram in which there are two distinct arcs, each overpassing a consecutive sequence of crossings, and every crossing in the diagram is in one of these sequences. The two arcs are called the bridges of the diagram (compare with [5], p. 23).

• numerator or denominator closures of rational tangles (see Figures 1, 5). A rational tangle is the result of consecutive twists on neighboring endpoints of two trivial arcs. For examples see Figure 1 and Figure 3.

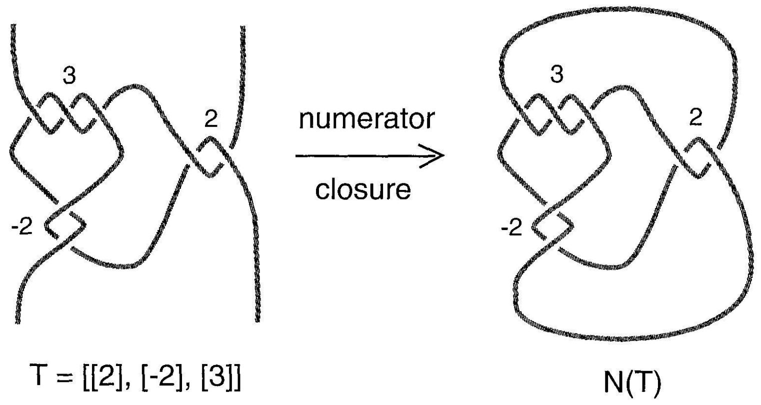

FIGURE 1
A rational tangle and a rational knot

All three representations are equivalent. The equivalence between the first and the third is easy to see by planar isotopies. For the equivalence between the first and the second representation see for example [5], pp. 23, 24. In this paper we consider rational knots as obtained by taking numerator or denominator closures of rational tangles (see Figure 5).

The notion of a tangle was introduced in 1967 by Conway [6] in his work on enumerating and classifying knots and links, and he defined the rational knots as numerator or denominator closures of the rational tangles. (It is worth noting here that Figure 2 in [1] illustrates a rational tangle, but no special importance is given to this object. It is obtained from a four-strand braid by plat-closing only the top four ends.) Conway [6] also defined *the fraction* of a rational tangle to be a rational number or ∞. He observed that this number for a rational tangle equals a continued fraction expression with all numerators equal to one and all denominators of the same sign, that can be read from a tangle diagram in alternating standard form. Rational tangles are classified by their fractions by means of the following theorem.

THEOREM 1 (Conway, 1975). *Two rational tangles are isotopic if and only if they have the same fraction.*

Proofs of Theorem 1 are given in [21], [5] p. 196, [13] and [15]. The first two proofs invoked the classification of rational knots and the theory of branched covering spaces. The 2-fold branched covering spaces of S^3 along the rational links give rise to the lens spaces $L(p,q)$. See [33] for a pioneering treatment of branched coverings. The proof in [13] is the first combinatorial proof of this theorem. The proofs in [21], [5] and [13] use definitions different from the above for the fraction of a rational tangle. In [15] a new combinatorial proof of Theorem 1 is given using the solution of the Tait Conjecture for alternating knots [42], [20] adapted for tangles. A second combinatorial proof is given in [15] using coloring for defining the tangle fraction.

Throughout the paper by the term 'knots' we will refer to both knots and links, and whenever we really mean 'knot' we shall emphasize it. More than one rational tangle can yield the same or isotopic rational knots and the equivalence relation between the rational tangles is mapped into an arithmetic equivalence of their corresponding fractions. Indeed we have the following

THEOREM 2 (Schubert, 1956). *Suppose that rational tangles with fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ are given (p and q are relatively prime; similarly for p' and q'). If $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ denote the corresponding rational knots obtained by taking numerator closures of these tangles, then $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ are isotopic if and only if*

1. $p = p'$ *and*
2. *either* $q \equiv q' \mod p$ *or* $q q' \equiv 1 \mod p$.

Schubert [31] originally stated the classification of rational knots and links by representing them as 2-bridge links. Theorem 2 has hitherto been proved by taking the 2-fold branched covering spaces of S^3 along 2-bridge links, showing that these correspond bijectively to oriented diffeomorphism classes of lens spaces, and invoking the classification of lens spaces [28]. Another proof using covering spaces has been given by Burde in [4]. See also the excellent notes on the subject by Siebenmann [35]. The above statement of Schubert's theorem is a formulation of the Theorem in the language of Conway's tangles.

Using his methods for the unoriented case, Schubert also extended the classification of rational knots and links to the case of oriented rational knots and links described as 2-bridge links. Here is our formulation of the Oriented Schubert Theorem written in the language of Conway's tangles.

THEOREM 3 (Schubert, 1956). *Suppose that orientation-compatible rational tangles with fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ are given with q and q' odd (p and q are relatively prime; similarly for p' and q'). If $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ denote the corresponding rational knots obtained by taking numerator closures of these tangles, then $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ are isotopic if and only if*

1. $p = p'$ *and*
2. *either* $q \equiv q' \mod 2p$ *or* $q q' \equiv 1 \mod 2p$.

Theorems 2 and 3 could have been stated equivalently using the denominator closures of rational tangles. Then the arithmetic equivalences of the tangle fractions related to isotopic knots would be the same as in Theorems 2 and 3, but with the roles of numerators and denominators exchanged.

This paper gives the first combinatorial proofs of Theorems 2 and 3 using tangle theory. Our proof of Theorem 2 uses the results and the techniques developed in [15], while the proof of Theorem 3 is based on that of Theorem 2. We have located the essential points in the proof of the classification of rational knots in the question: *Which rational tangles will close to form a specific knot or link diagram?* By looking at the Theorems in this way, we obtain a path to the results that can be understood without extensive background in three-dimensional topology. In the course of these proofs we see connections between the elementary number theory of fractions and continued fractions, and the topology of knots and links. In order to compose these proofs we use the fact that rational knots are alternating (which follows from the fact that rational tangles are alternating, and for which we believe we found the simplest possible proof, see [15], Proposition 2). We then rely on the *Tait Conjecture* [42] concerning the classification of alternating knots, which states the following:

Two alternating knots are isotopic if and only if any two corresponding reduced diagrams on S^2 are related by a finite sequence of flypes (see Figure 6).

A diagram is said to be *reduced* if at every crossing the four local regions indicated at the crossing are actually parts of four distinct global regions in the diagram (see [19], p. 42). It is not hard to see that any knot or link has reduced diagrams that represent its isotopy class. The conjecture was posed by P. G. Tait [42] in 1877 and was proved by W. Menasco and M. Thistlethwaite, [20] in 1993. Tait did not actually phrase this statement as a conjecture. It was a working hypothesis for his efforts in classifying knots.

Our proof of the Schubert Theorem is elementary upon assuming the Tait Conjecture, but this is easily stated and understood. This paper will be of interest to mathematicians and biologists.

The paper is organized as follows. In Section 2 we give the general set up for rational tangles, their isotopies and operations, as well as their association to a continued fraction isotopy invariant. In this section we also recall the basic theory and a canonical form of continued fractions. In Section 3 we prove Theorem 2 about the classification of unoriented rational knots by means of a direct combinatorial and arithmetical analysis of rational knot diagrams, using the classification of rational tangles and the Tait Conjecture. In Section 4 we discuss chirality of knots and give a classification of the achiral rational knots and links as numerator closures of even palindromic rational tangles in continued fraction form (Theorem 5). In Section 5 we discuss the connectivity patterns of the four end arcs of rational tangles and we relate connectivity to the parity of the fraction of a rational tangle (Theorem 6). In Section 6 we give our interpretation of the statement of Theorem 3 and we prove the classification of oriented rational knots, using the methods we developed in the unoriented case and examining the connectivity patterns of oriented rational knots. In Section 6 it is pointed out that all oriented rational knots and links are invertible (reverse the orientation of both components). In Section 7 we give a classification of the strongly invertible rational links (reverse the orientation of one component) as closures of odd palindromic oriented rational tangles in continued fraction form (Theorem 7).

Here is a short history of the theory of rational knots. As explained in [14], rational knots and links were first considered by O. Simony in 1882, [36, 37, 38, 39], taking twistings and knottings of a band. Simony [37] was the first one to relate knots to continued fractions. After about sixty years Tietze wrote a series of papers [43, 44, 45, 46] with reference to Simony's work. Reidemeister [27] in 1929 calculated the knot group of a special class of four-plats (Viergeflechte), but four-plats were really studied by Goeritz [12] and by Bankwitz and Schumann [1] in 1934. In [12] and [1] proofs are given independently and with different techniques that rational knots have 3-strand-braid representations, in the sense that the first strand of the four-strand braids can be free of crossings, and that they are alternating. (See Figure 20 for an example and Figure 26 for an abstract 3-strand-braid representation.) The proof of the latter in [1] can be easily applied on the corresponding rational tangles in standard form. (See Figure 1 for an example and Figure 8 for abstract representations.)

In 1954 Schubert [30] introduced the bridge representation of knots. He then showed that the four-plats are exactly the knots that can be represented by diagrams with two bridges and consequently he classified rational knots by finding canonical forms via representing them as 2-bridge knots, see [31]. His proof was based on Seifert's observation that the 2-fold branched coverings of 2-bridge knots [33] give rise to lens spaces and on the classification of lens spaces by Reidemeister [28] using Reidemeister torsion and following the lead of [32] (and later by Brody [3] using the knot theory of the lens space). See also [25]. Rational knots and rational tangles figure prominently in the applications of knot theory to the topology of DNA, see [40]. Treatments of various aspects of rational knots and rational tangles can be found in many places in the literature, see for example [6], [35], [29], [5], [2], [22], [16], [19].

2. Rational tangles and their invariant fractions

In this section we recall from [15] the facts that we need about rational tangles, continued fractions and the classification of rational tangles. We intend the paper to be as self-contained as possible.

A 2*-tangle* is a proper embedding of two unoriented arcs and a finite number of circles in a 3-ball B^3, so that the four endpoints lie in the boundary of B^3. A *rational tangle* is a proper embedding of two unoriented arcs α_1, α_2 in a 3-ball B^3, so that the four endpoints lie in the boundary of B^3, and such that there exists a homeomorphism of pairs:

$$\overline{h}\colon (B^3, \alpha_1, \alpha_2) \to (D^2 \times I,\ \{x, y\} \times I) \quad \text{(a trivial tangle)}.$$

This is equivalent to saying that rational tangles have specific representatives obtained by applying a finite number of consecutive twists of neighboring endpoints starting from two unknotted and unlinked arcs. Such a pair of arcs comprise the [0] or [∞] tangles, depending on their position in the plane, see illustrations in Figure 2.

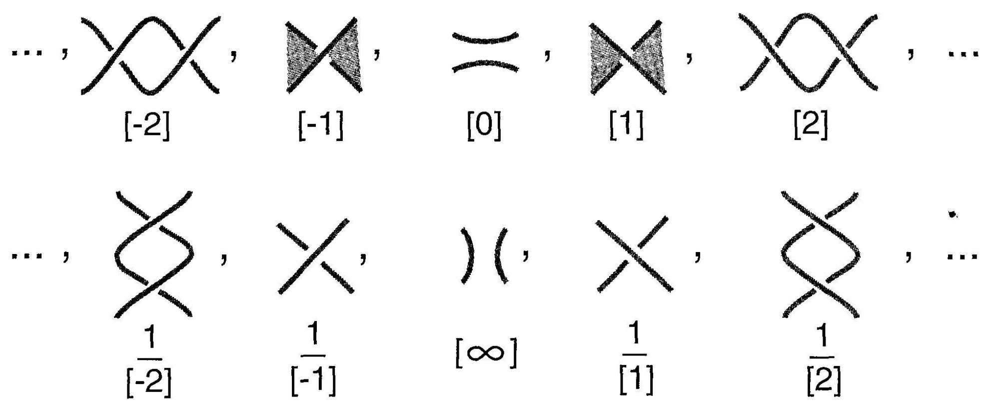

FIGURE 2
The elementary rational tangles and the types of crossings

We shall use this characterizing property of a rational tangle as our definition, and we shall then say that the rational tangle is in *twist form*. See Figure 3 for an example.