2.1 Hamiltonian mechanics and integrable Systems

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From this we deduce

THEOREM 1.14 ([EG2, BEG, FeV], conjectured in [FV]). *The ring Q_m of m-quasi-invariants is Gorenstein.*

Proof. By Stanley's theorem (see [Eis]), a positively graded Cohen-Macaulay domain A is Gorenstein iff its Poincaré series is a rational function $h(t)$ satisfying the equation $h(t^{-1}) = (-1)^n t^l h(t)$, where l is an integer and n is the dimension of the spectrum of A. Thus the result follows immediately from Proposition 1.13. □

1.6 THE RING OF DIFFERENTIAL OPERATORS ON X_m

Finally, let us introduce the ring $\mathcal{D}(X_m)$ of differential operators on X_m, that is the ring of differential operators with coefficients in $\mathbf{C}(\mathfrak{h})$ mapping Q_m to Q_m. It is clear that this definition coincides with Grothendieck's well-known definition ([Bj]).

THEOREM 1.15 ([BEG]). *$\mathcal{D}(X_m)$ is a simple algebra.*

REMARK 1.16. a) The ring of differential operators on a smooth affine algebraic variety is always simple (see [Bj], Chapter 3).

b) By a result of M. van den Bergh [VdB], for a non-smooth variety, the simplicity of the ring of differential operators implies the Cohen-Macaulay property of this variety.

2. LECTURE 2

We will now see how the ring Q_m appears in the theory of completely integrable systems.

2.1 HAMILTONIAN MECHANICS AND INTEGRABLE SYSTEMS

Recall the basic setup of Hamiltonian mechanics [Ar]. Consider a mechanical system with configuration space X (a smooth manifold). Then the phase space of this system is T^*X, the cotangent bundle on X. The space T^*X is naturally a symplectic manifold, and in particular we have an operation of Poisson bracket on functions on T^*X. A point of T^*X is a pair (x,p), where $x \in X$ is the position and $p \in T_x^*X$ is the momentum. Such pairs are called states of the system. The dynamics of the system $x = x(t)$, $p = p(t)$ depends on the Hamiltonian, or energy function, $E(x,p)$ on T^*X. Given E and the initial state $x(0)$, $p(0)$, one can recover the dynamics $x = x(t)$, $p = p(t)$ from Hamilton's differential equations $\frac{df(x,p)}{dt} = \{f, E\}$. If X is locally identified with $\mathbf{R}^n$ by choosing coordinates $x_1, \ldots, x_n$, then T^*X is locally identified with $\mathbf{R}^{2n}$ with coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$. In these coordinates, Hamilton's equations may be written in their standard form

$$\dot{x}_i = \frac{\partial E}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial E}{\partial x_i}.$$

A function $I(x,p)$ is called an integral of motion for our system if $\{I, E\} = 0$. Integrals of motion are useful, since for any such integral I the function $I(x(t), p(t))$ is constant, which allows one to reduce the number of variables by 2. Thus, if we are given n functionally independent integrals of motion $I_1, \ldots, I_n$ with $\{I_l, I_k\} = 0$ for all $1 \leq l, k \leq n$, then all $2n$ variables x_i, p_i can be excluded, and the system can be completely solved by quadratures. Such a situation is called complete (or Liouville) integrability.

2.2 The classical Calogero-Moser system

Quasi-invariants are related to many-particle systems. Consider a system of n particles on the real line $\mathbf{R}$. A potential is an even function

$$U(x) = U(-x), \quad x \in \mathbf{R}.$$

Two particles at points a, b have energy of interaction $U(a-b)$. The total energy of our system of particles is

$$E = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i<j} U(x_i - x_j).$$

Here, x_i are the coordinates of the particles, p_i their momenta. The dynamics of the particles $x_i = x_i(t)$, $p_i = p_i(t)$ is governed by the Hamilton equations with energy function E.

This is a system of nonlinear differential equations, which in general can be difficult to solve explicitly. However, for special potentials this system might be completely integrable. For instance, we will see that this is the case for the Calogero-Moser potential,

$$U(x) = \frac{\gamma}{x^2},$$

γ being a constant.