The Calogero-Moser system has a generalization to arbitrary Coxeter groups. Namely, consider a finite group W generated by reflections acting on the space $\mathfrak{h}$, and keep the notation of the previous section. Fix a W-invariant nondegenerate scalar product $(-,-)$ on $\mathfrak{h}$. It determines a scalar product on $\mathfrak{h}^*$. Define the "energy function"

$$E(x,p) = \frac{(p,p)}{2} + \frac{1}{2}\sum_{s\in\Sigma}\frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s(x)^2}, \quad x\in\mathfrak{h}, \quad p\in\mathfrak{h}^*$$

on $T^*\mathfrak{h} = \mathfrak{h}\times\mathfrak{h}^*$, where $\gamma\colon \Sigma\to\mathbf{C}$ is a W-invariant function. Notice that although α_s is defined up to a non zero constant, by homogeneity, E is independent of the choice of α_s. We will call the system defined by E the Calogero-Moser system for W.

If W is the symmetric group S_n, $\mathfrak{h} = \mathbf{C}^n$, then Σ is the set of transpositions $s_{i,j}$, $i<j$, and we can take $\alpha_s = e_i - e_j$, Then we clearly obtain the usual Calogero-Moser system.

Below we will see that the Calogero-Moser system for W is completely integrable.

2.3 The quantum Calogero-Moser system

Let us now discuss quantization of the Calogero-Moser system. We start by quantizing the energy E by formally making the substitution

$$p_j \Rightarrow -i\hbar\frac{\partial}{\partial x_j},$$

where $\hbar$ is a parameter (Planck's constant). This yields the Schrödinger operator

$$\widehat{E} := -\frac{\hbar^2}{2}\Delta + \frac{1}{2}\sum_{s\in\Sigma}\frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s^2},$$

where Δ denotes the Laplacian.

In particular, in the case of $W = S_n$ we have

$$\widehat{E} = -\frac{\hbar^2}{2}\Delta + \sum_{i<j}\frac{c}{(x_i-x_j)^2},$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Setting $\beta_s = \frac{\gamma_s}{2\hbar^2}$, we will from now on consider the operator

$$H := -\frac{2}{\hbar^2}\widehat{E} = \Delta - \sum_{s\in\Sigma}\frac{\beta_s(\alpha_s,\alpha_s)}{\alpha_s^2(x)},$$

called the Calogero-Moser operator.

We want to study the stationary Schrödinger equation:

$$H\psi = \lambda\psi, \quad \lambda \in \mathbf{C}. \tag{3}$$

As in the classical case, it is difficult to say anything explicit about solutions of this equation for a general Schrödinger operator H, but for the Calogero-Moser operator the situation is much better.

DEFINITION 2.1. A *quantum integral* of H is a differential operator M such that

$$[M,H] = 0.$$

We are going to show that there are many quantum integrals of H, namely that there are n commuting algebraically independent quantum integrals $M_1, \dots, M_n$ of H. By definition, this means that the quantum Calogero-Moser system is completely integrable.

Once we have found $M_1, \dots, M_n$, observe that for fixed constants $\mu_1, \dots, \mu_n$, the space of solutions of the system

$$\begin{cases} M_1\psi = \mu_1\psi \\ \quad\cdots\cdots \\ M_n\psi = \mu_n\psi \end{cases}$$

is clearly stable under H. We will see that this space is in fact finite dimensional. Therefore, the operators M_i allow one to reduce the problem of solving the partial differential equation $H\psi = \lambda\psi$ to that of solving a system of ordinary linear differential equations. This phenomenon is called quantum complete integrability.

2.4 THE ALGEBRA OF DIFFERENTIAL-REFLECTION OPERATORS

We are now going to explain how to find quantum integrals for H, using the Dunkl-Cherednik method.

First let us fix some notation. Given a smooth affine variety X, we will denote by $\mathcal{D}(X)$ the ring of differential operators on X. We are going to consider the case in which X is the open set U in $\mathfrak{h}$ which is the complement of the divisor of the equation $\delta(x) := \prod_{s\in\Sigma} \alpha_s(x)$. Clearly $\mathcal{D}(U) = \mathcal{D}(\mathfrak{h})[1/\delta(x)]$.

LEMMA 2.2. *An element of $\mathcal{D}(U)$ is completely determined by its action on $\mathbf{C}[U]^W = \mathbf{C}[U/W]$.*