

## 2.6 Additional integrals for integer valued $c$

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$d_i$  being the degrees of basic  $W$ -invariants, we obtain a polynomial ring of commuting differential operators in  $\mathcal{D}(U)$ . Given  $q \in \mathbf{C}[q_1, \dots, q_n]$  we will denote by  $L_q$  the corresponding differential operator. We may assume that  $q_1 = \sum_{i=1}^n y_i^2$  so that  $L = L_{q_1}$ . Thus for every  $q \in \mathbf{C}[q_1, \dots, q_n]$ ,  $L_q$  is a quantum integral of the quantum Calogero-Moser system. In particular, the operators  $L_{q_1}, \dots, L_{q_n}$  are  $n$  algebraically independent pairwise commuting quantum integrals.

Now the eigenvalue problem (4) may be replaced by

$$L_p \psi = \lambda_p \psi$$

for  $p \in \mathbf{C}[q_1, \dots, q_n]$ , where the assignment  $p \rightarrow \lambda_p$  is an algebra homomorphism  $\mathbf{C}[q_1, \dots, q_n] \rightarrow \mathbf{C}$ .

In other words, we may say that since  $\mathbf{C}[q_1, \dots, q_n] = \mathbf{C}[\mathfrak{h}^*/W] = \mathbf{C}[\mathfrak{h}/W]$ , for every point  $k \in \mathfrak{h}/W$ , we have the eigenvalue problem

$$(5) \quad L_p \psi = p(k) \psi.$$

**PROPOSITION 2.8.** *Near a generic point  $x_0 \in \mathfrak{h}$ , the system  $L_p \psi = p(k) \psi$  has a space of solutions of dimension  $|W|$ .*

*Proof.* The proposition follows easily from the fact that the symbols of  $L_{q_i}$  are  $q_i(\partial)$ , and that  $\mathbf{C}[y_1, \dots, y_n]$  is a free module over  $\mathbf{C}[q_1, \dots, q_n]$  of rank  $|W|$ .  $\square$

## 2.6 ADDITIONAL INTEGRALS FOR INTEGER VALUED $c$

If  $c_s \notin \mathbf{Z}$ , the analysis of the solutions of the equations  $L_p \psi = p(k) \psi$  is rather difficult (see [HO]). However, in the case  $c: \Sigma \rightarrow \mathbf{Z}$ , the system can be simplified. Let us consider this case. First remark that, since  $\beta_s = c_s(c_s + 1)$ , by changing  $c_s$  to  $-1 - c_s$  if necessary, we may assume that  $c$  is non-negative. So we will assume that  $c$  takes non-negative integral values and we will denote it by  $m$ .

System (5) can be further simplified, if we can find a differential operator  $M$  (not a polynomial of  $L_{q_1}, \dots, L_{q_n}$ ) such that  $[M, L_p] = 0$  for all  $p \in \mathbf{C}[q_1, \dots, q_n]$ . Then the operator  $M$  will act on the space of solutions of (5), hopefully with distinct eigenvalues. So if  $\mu$  is such an eigenvalue, the system

$$\begin{cases} L_p \psi = p(k) \psi \\ M \psi = \mu \psi \end{cases}$$

will have a one dimensional space of solutions and we can find the unique up to scaling solution  $\psi$  using Euler's formula.

Such an  $M$  exists if and only if  $c = m$  has integer values. Namely, we will see that one can extend the homomorphism  $\mathbf{C}[q_1, \dots, q_n] \rightarrow \mathcal{D}(U)$  mapping  $q \rightarrow L_q$  to the ring of  $m$ -quasi-invariants  $Q_m$ .

We start by remarking that under some natural homogeneity assumptions, if such an extension exists, it is unique.

PROPOSITION 2.9. 1) Assume that  $q \in \mathbf{C}[y_1, \dots, y_n]$  is a homogeneous polynomial of degree  $d$ . If there exists a differential operator  $M_q$  with coefficients in  $\mathbf{C}(h)$ , of the form

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

such that  $[M_q, L] = 0$ , whose homogeneity degree is  $-d$ , then  $M_q$  is unique.

2) Let  $\mathbf{C}[q_1, \dots, q_n] \subseteq B \subseteq \mathbf{C}[y_1, \dots, y_n]$  be a graded ring. Assume that we have a linear map  $M: B \rightarrow \mathcal{D}(U)$  such that, if  $q \in B$  is homogeneous of degree  $d$ , then  $[M_q, L] = 0$ ,  $M_q$  has homogeneity degree  $-d$ , and

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

Then  $M$  is a ring homomorphism and  $M_q = L_q$  for all  $q \in \mathbf{C}[q_1, \dots, q_n]$ .

*Proof.* 1) If there exist two different operators  $M_q$  and  $M'_q$  with these properties, take  $M_q - M'_q$ . This operator has degree of homogeneity  $-d$ , but order smaller than  $d$ . Therefore, its symbol  $S(x, y)$  is not a polynomial. On the other hand, since the symbol of  $L$  is  $\sum y_i^2$ , we get that  $[L, M_q - M'_q] = 0$  implies  $\{\sum y_i^2, S(x, y)\} = 0$ . Write  $S$  in the form  $K(x, y)/H(x)$  with  $K$  is a polynomial, and  $H(x)$  a homogeneous polynomial of positive degree  $t$  (we assume that  $K(x, y)$  and  $H(x)$  have no common irreducible factors). Then

$$0 = \left\{ \sum y_i^2, S(x, y) \right\} = 2 \frac{\sum_{i=1}^n y_i K_{x_i}(x, y) H(x) - \sum_{i=1}^n y_i H_{x_i}(x) K(x, y)}{H(x)^2}.$$

Since  $\sum_{i=1}^n x_i H_{x_i}(x) = tH(x)$ , we have  $\sum_{i=1}^n y_i H_{x_i}(x) K(x, y) \neq 0$ . So  $H(x)$  must divide this polynomial and, by our assumptions, this implies that it must divide the polynomial  $\sum_{i=1}^n y_i H_{x_i}(x)$  whose degree in  $x$  is  $t - 1$ . This is a contradiction.

2) Let  $q, p \in B$  be two homogeneous elements. Then  $M_q M_p$  and  $M_{pq}$  both satisfy the same homogeneity assumptions. Hence they are equal by 1).

Finally if  $q \in \mathbf{C}[q_1, \dots, q_n]$ , both  $M_q$  and  $L_q$  satisfy the same homogeneity assumptions. Hence they are equal by 1).  $\square$

The required extension to the ring of  $m$ -quasi-invariants is then provided by the following

**THEOREM 2.10** ([CV1, CV2]). *Let  $c = m: \Sigma \rightarrow \mathbf{Z}_+$ . The following two conditions are equivalent for a homogeneous polynomial  $q \in \mathbf{C}[\mathfrak{h}^*]$  of degree  $d$ .*

1) *There exists a differential operator*

$$L_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

*of homogeneity degree  $-d$ , such that  $[L_q, L] = 0$ .*

2)  *$q$  is an  $m$ -quasi-invariant homogeneous of degree  $d$ .*

Using this, we can extend system (5) to the system

$$(6) \quad L_p \psi = p(k)\psi, \quad p \in Q_m, \quad k \in \text{Spec } Q_m = X_m.$$

(Recall that, as a set,  $X_m = \mathfrak{h}$ .) Near a generic point  $x_0 \in \mathfrak{h}$ , system (6) has a one dimensional space of solutions, thus there exists a unique up to scaling solution  $\psi(k, x)$ , which can be expressed in elementary functions. This solution is called the *Baker-Akhiezer function*, and has the form

$$\psi(k, x) = P(k, x) e^{(k, x)}$$

with  $P(k, x)$  a polynomial of the form  $\delta(x)\delta(k) + l.o.t.$  and  $e^{(k, x)}$  denotes the exponential function computed in the scalar product  $(k, x)$ . Furthermore, it can be shown that  $\psi(k, x) = \psi(x, k)$  (see [CV1, CV2, FV]).

These results motivate the following terminology. The variety  $X_m$  is called *the spectral variety* of the Calogero-Moser system for the multiplicity function  $m$ , and  $Q_m$  is called *the spectral ring* of this system.

## 2.7 AN EXAMPLE

**EXAMPLE 2.11.** Let  $W = \mathbf{Z}/2$ ,  $\mathfrak{h} = \mathbf{C}$ ,  $m = 1$ . As we have seen,  $Q_m$  has a basis given by the monomials  $\{x^{2i}\} \cup \{x^{2i+3}\}$ ,  $i \geq 0$ . Let us set for such a monomial,  $L_{x^r} = L_r$ , and  $\partial = \frac{d}{dx}$ . Then we have

$$L_0 = 1, \quad L_2 = \partial^2 - \frac{2}{x}\partial, \quad L_3 = \partial^3 - \frac{3}{x}\partial^2 + \frac{3}{x^2}\partial.$$

As for the others,  $L_{2j} = L_2^j$ ,  $L_{2j+3} = L_2^j L_3$ . (Note that  $L_1$  is not defined). The system (6) in this case is

$$\begin{cases} \psi'' - \frac{2}{x}\psi' = k^2\psi, \\ \psi''' - \frac{3}{x}\psi'' + \frac{3}{x^2}\psi' = k^3\psi. \end{cases}$$