2.6 Additional integrals for integer valued c

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d_i being the degrees of basic W-invariants, we obtain a polynomial ring of commuting differential operators in $\mathcal{D}(U)$. Given $q \in \mathbf{C}[q_1, \ldots, q_n]$ we will denote by L_q the corresponding differential operator. We may assume that $q_1 = \sum_{i=1}^n y_1^2$ so that $L = L_{q_1}$. Thus for every $q \in \mathbf{C}[q_1, \ldots, q_n]$, L_q is a quantum integral of the quantum Calogero-Moser system. In particular, the operators $L_{q_1}, \ldots, L_{q_n}$ are n algebraically independent pairwise commuting quantum integrals.

Now the eigenvalue problem (4) may be replaced by

$$L_p\psi = \lambda_p\psi$$

for $p \in \mathbf{C}[q_1, \ldots, q_n]$, where the assignment $p \to \lambda_p$ is an algebra homomorphism $\mathbf{C}[q_1, \ldots, q_n] \to \mathbf{C}$.

In other words, we may say that since $\mathbf{C}[q_1, \ldots, q_n] = \mathbf{C}[\mathfrak{h}^*/W] = \mathbf{C}[\mathfrak{h}/W]$, for every point $k \in \mathfrak{h}/W$, we have the eigenvalue problem

$$L_p\psi = p(k)\psi\,. \tag{5}$$

PROPOSITION 2.8. *Near a generic point $x_0 \in \mathfrak{h}$, the system $L_p\psi = p(k)\psi$ has a space of solutions of dimension $|W|$.*

Proof. The proposition follows easily from the fact that the symbols of L_{q_i} are $q_i(\partial)$, and that $\mathbf{C}[y_1, \ldots, y_n]$ is a free module over $\mathbf{C}[q_1, \ldots, q_n]$ of rank $|W|$. □

2.6 ADDITIONAL INTEGRALS FOR INTEGER VALUED c

If $c_s \notin \mathbf{Z}$, the analysis of the solutions of the equations $L_p\psi = p(k)\psi$ is rather difficult (see [HO]). However, in the case $c\colon \Sigma \to \mathbf{Z}$, the system can be simplified. Let us consider this case. First remark that, since $\beta_s = c_s(c_s+1)$, by changing c_s to $-1-c_s$ if necessary, we may assume that c is non-negative. So we will assume that c takes non-negative integral values and we will denote it by m.

System (5) can be further simplified, if we can find a differential operator M (not a polynomial of $L_{q_1}, \ldots, L_{q_n}$) such that $[M, L_p] = 0$ for all $p \in \mathbf{C}[q_1, \ldots, q_n]$. Then the operator M will act on the space of solutions of (5), hopefully with distinct eigenvalues. So if μ is such an eigenvalue, the system

$$\begin{cases} L_p\psi = p(k)\,\psi \\ M\psi = \mu\,\psi \end{cases}$$

will have a one dimensional space of solutions and we can find the unique up to scaling solution ψ using Euler's formula.

Such an M exists if and only if $c = m$ has integer values. Namely, we will see that one can extend the homomorphism $\mathbf{C}[q_1, \dots, q_n] \to \mathcal{D}(U)$ mapping $q \to L_q$ to the ring of m-quasi-invariants Q_m.

We start by remarking that under some natural homogeneity assumptions, if such an extension exists, it is unique.

PROPOSITION 2.9. 1) *Assume that* $q \in \mathbf{C}[y_1, \dots, y_n]$ *is a homogeneous polynomial of degree* d. *If there exists a differential operator* M_q *with coefficients in* $\mathbf{C}(\mathfrak{h})$, *of the form*

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

such that $[M_q, L] = 0$, *whose homogeneity degree is* $-d$, *then* M_q *is unique.*

2) *Let* $\mathbf{C}[q_1, \dots, q_n] \subseteq B \subseteq \mathbf{C}[y_1, \dots, y_n]$ *be a graded ring. Assume that we have a linear map* $M\colon B \to \mathcal{D}(U)$ *such that, if* $q \in B$ *is homogeneous of degree* d, *then* $[M_q, L] = 0$, M_q *has homogeneity degree* $-d$, *and*

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

Then M *is a ring homomorphism and* $M_q = L_q$ *for all* $q \in \mathbf{C}[q_1, \dots, q_n]$.

Proof. 1) If there exist two different operators M_q and M'_q with these properties, take $M_q - M'_q$. This operator has degree of homogeneity $-d$, but order smaller than d. Therefore, its symbol $S(x, y)$ is not a polynomial. On the other hand, since the symbol of L is $\sum y_i^2$, we get that $[L, M_q - M'_q] = 0$ implies $\{\sum y_i^2, S(x, y)\} = 0$. Write S in the form $K(x, y)/H(x)$ with K is a polynomial, and $H(x)$ a homogeneous polynomial of positive degree t (we assume that $K(x, y)$ and $H(x)$ have no common irreducible factors). Then

$$0 = \{\sum y_i^2, S(x, y)\} = 2\,\frac{\sum_{i=1}^n y_i K_{x_i}(x, y) H(x) - \sum_{i=1}^n y_i H_{x_i}(x) K(x, y)}{H(x)^2}\,.$$

Since $\sum_{i=1}^n x_i H_{x_i}(x) = tH(x)$, we have $\sum_{i=1}^n y_i H_{x_i}(x) K(x, y) \neq 0$. So $H(x)$ must divide this polynomial and, by our assumptions, this implies that it must divide the polynomial $\sum_{i=1}^n y_i H_{x_i}(x)$ whose degree in x is $t - 1$. This is a contradiction.

2) Let $q, p \in B$ be two homogeneous elements. Then $M_q M_p$ and M_{pq} both satisfy the same homogeneity assumptions. Hence they are equal by 1).

Finally if $q \in \mathbf{C}[q_1, \dots, q_n]$, both M_q and L_q satisfy the same homogeneity assumptions. Hence they are equal by 1). □

The required extension to the ring of m-quasi-invariants is then provided by the following

THEOREM 2.10 ([CV1, CV2]). *Let $c = m: \Sigma \to \mathbf{Z}_+$. The following two conditions are equivalent for a homogeneous polynomial $q \in \mathbf{C}[\mathfrak{h}^*]$ of degree d.*

1) *There exists a differential operator*

$$L_q = q(\partial_{y_1}, \ldots, \partial_{y_n}) + l.o.t.$$

of homogeneity degree $-d$, such that $[L_q, L] = 0$.

2) *q is an m-quasi-invariant homogeneous of degree d.*

Using this, we can extend system (5) to the system

$$L_p\psi = p(k)\psi\,, \quad p \in Q_m\,, \quad k \in \operatorname{Spec} Q_m = X_m\,. \tag{6}$$

(Recall that, as a set, $X_m = \mathfrak{h}$.) Near a generic point $x_0 \in \mathfrak{h}$, system (6) has a one dimensional space of solutions, thus there exists a unique up to scaling solution $\psi(k,x)$, which can be expressed in elementary functions. This solution is called the *Baker-Akhiezer function*, and has the form

$$\psi(k,x) = P(k,x)\, e^{(k,x)}$$

with $P(k,x)$ a polynomial of the form $\delta(x)\delta(k) + l.o.t.$ and $e^{(k,x)}$ denotes the exponential function computed in the scalar product (k,x). Furthermore, it can be shown that $\psi(k,x) = \psi(x,k)$ (see [CV1, CV2, FV]).

These results motivate the following terminology. The variety X_m is called *the spectral variety* of the Calogero-Moser system for the multiplicity function m, and Q_m is called *the spectral ring* of this system.

2.7 AN EXAMPLE

EXAMPLE 2.11. Let $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, $m = 1$. As we have seen, Q_m has a basis given by the monomials $\{x^{2i}\} \cup \{x^{2i+3}\}$, $i \geq 0$. Let us set for such a monomial, $L_{x^r} = L_r$, and $\partial = \frac{d}{dx}$. Then we have

$$L_0 = 1, \quad L_2 = \partial^2 - \frac{2}{x}\partial, \quad L_3 = \partial^3 - \frac{3}{x}\partial^2 + \frac{3}{x^2}\partial\,.$$

As for the others, $L_{2j} = L_2^j$, $L_{2j+3} = L_2^j L_3$. (Note that L_1 is not defined). The system (6) in this case is

$$\begin{cases} \psi'' - \dfrac{2}{x}\psi' = k^2\psi\,, \\ \psi''' - \dfrac{3}{x}\psi'' + \dfrac{3}{x^2}\psi' = k^3\psi\,. \end{cases}$$