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The required extension to the ring of  $m$ -quasi-invariants is then provided by the following

**THEOREM 2.10** ([CV1, CV2]). *Let  $c = m: \Sigma \rightarrow \mathbf{Z}_+$ . The following two conditions are equivalent for a homogeneous polynomial  $q \in \mathbf{C}[\mathfrak{h}^*]$  of degree  $d$ .*

1) *There exists a differential operator*

$$L_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

*of homogeneity degree  $-d$ , such that  $[L_q, L] = 0$ .*

2)  *$q$  is an  $m$ -quasi-invariant homogeneous of degree  $d$ .*

Using this, we can extend system (5) to the system

$$(6) \quad L_p \psi = p(k)\psi, \quad p \in Q_m, \quad k \in \text{Spec } Q_m = X_m.$$

(Recall that, as a set,  $X_m = \mathfrak{h}$ .) Near a generic point  $x_0 \in \mathfrak{h}$ , system (6) has a one dimensional space of solutions, thus there exists a unique up to scaling solution  $\psi(k, x)$ , which can be expressed in elementary functions. This solution is called the *Baker-Akhiezer function*, and has the form

$$\psi(k, x) = P(k, x) e^{(k, x)}$$

with  $P(k, x)$  a polynomial of the form  $\delta(x)\delta(k) + l.o.t.$  and  $e^{(k, x)}$  denotes the exponential function computed in the scalar product  $(k, x)$ . Furthermore, it can be shown that  $\psi(k, x) = \psi(x, k)$  (see [CV1, CV2, FV]).

These results motivate the following terminology. The variety  $X_m$  is called *the spectral variety* of the Calogero-Moser system for the multiplicity function  $m$ , and  $Q_m$  is called *the spectral ring* of this system.

## 2.7 AN EXAMPLE

**EXAMPLE 2.11.** Let  $W = \mathbf{Z}/2$ ,  $\mathfrak{h} = \mathbf{C}$ ,  $m = 1$ . As we have seen,  $Q_m$  has a basis given by the monomials  $\{x^{2i}\} \cup \{x^{2i+3}\}$ ,  $i \geq 0$ . Let us set for such a monomial,  $L_{x^r} = L_r$ , and  $\partial = \frac{d}{dx}$ . Then we have

$$L_0 = 1, \quad L_2 = \partial^2 - \frac{2}{x}\partial, \quad L_3 = \partial^3 - \frac{3}{x}\partial^2 + \frac{3}{x^2}\partial.$$

As for the others,  $L_{2j} = L_2^j$ ,  $L_{2j+3} = L_2^j L_3$ . (Note that  $L_1$  is not defined). The system (6) in this case is

$$\begin{cases} \psi'' - \frac{2}{x}\psi' = k^2\psi, \\ \psi''' - \frac{3}{x}\psi'' + \frac{3}{x^2}\psi' = k^3\psi. \end{cases}$$

The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \frac{2}{x}\psi' = k^2\psi, \\ \psi'' - (\frac{1}{x} + k^2x)\psi' = -k^3x\psi. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2x}{kx - 1}\psi,$$

whose solution (up to constants) is given by  $\psi = (kx - 1)e^{kx}$ .

In fact, one can easily calculate  $\psi_m$  for a general  $m$ .

PROPOSITION 2.12.  $\psi_m(k, x) = (x\partial - 2m + 1)(x\partial - 2m - 1) \cdots (x\partial - 1)e^{kx}$ .

*Proof.* We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$(\partial^2 - \frac{2m}{x}\partial)(x\partial - 2m + 1) = (x\partial - 2m + 1)(\partial^2 - \frac{2(m-1)}{x}\partial)$$

as it is easy to verify directly. So using induction on  $m$  starting with  $m = 0$ , we get

$$(\partial^2 - \frac{2m}{x}\partial)\psi_m(k, x) = (x\partial - 2m + 1)(\partial^2 - \frac{2(m-1)}{x}\partial)\psi_{m-1}(k, x) = k^2\psi_m(k, x),$$

and  $\psi_m(k, x)$  is our solution.  $\square$

### 3. LECTURE 3

#### 3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function  $\psi(k, x)$  for the operator

$$L = \Delta - \sum_{s \in \Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s}.$$

The way to construct  $\psi(k, x)$  is via the Opdam shift operator. Given a function  $m: \Sigma \rightarrow \mathbf{Z}_+$ , Opdam showed in [Op1] that there exists a unique  $W$ -invariant