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Kapitel: 3.4 The Cherednik algebra

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3.4 THE CHEREDNIK ALGEBRA

Let us now return to the algebra $\mathcal{A}$ of operators on U generated by $\mathcal{D}(U)$ and W. This algebra contains the Dunkl operators

$$D_y := \partial_y + \sum_{s\in\Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s}(s-1).$$

LEMMA 3.10. *The following relations hold:*

$$[x_i, x_j] = [D_{x_i}, D_{x_j}] = 0, \quad \forall 1 \le i,j \le n$$

$$[D_{x_i}, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i, \alpha_s)(x_j, \alpha_s)}{(\alpha_s, \alpha_s)} s, \quad \forall 1 \le i,j \le n$$

$$wxw^{-1} = w(x), \quad wD_y w^{-1} = D_{w(y)}, \quad \forall w \in W,\ x \in \mathfrak{h}^*,\ y \in \mathfrak{h}.$$

Proof. The proof is an easy computation, except for the relations $[D_{x_i}, D_{x_j}] = 0$, which follow from Theorem 2.6. □

This lemma motivates the following definition.

DEFINITION 3.11 (see e.g. [EG]). The *Cherednik algebra* H_c is an associative algebra with generators $x_i, y_i, i = 1, \dots, n$, and $w \in W$, with defining relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad \forall 1 \le i,j \le n$$

$$[y_i, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i, \alpha_s)(x_j, \alpha_s)}{(\alpha_s, \alpha_s)} s, \quad \forall 1 \le i,j \le n$$

$$wxw^{-1} = w(x),\ wyw^{-1} = w(y),\ w \cdot w' = ww',\ \forall w, w' \in W,\ x \in \mathfrak{h}^*,\ y \in \mathfrak{h}.$$

This algebra was introduced by Cherednik as a rational limit of his double affine Hecke algebra defined in [Ch]. Notice that if $c = 0$ then $H_0 = \mathcal{D}(\mathfrak{h}) \rtimes \mathbf{C}[W]$.

Lemma 3.10 implies that the algebra H_c is equipped with a homomorphism $\phi\colon H_c \to \mathcal{A}$, given by $w \to w$, $x_i \to x_i$, $y_i \to D_{x_i}$.

Cherednik proved the following theorem.

THEOREM 3.12 (Poincaré-Birkhoff-Witt theorem). *The multiplication map*

$$\mu\colon \mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[W] \to H_c$$

given by $\mu(f(x) \otimes g(y) \otimes w) = f(x)\, g(y) w$ *is an isomorphism of vector spaces.*

Proof. It is easy to see that the map μ is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials $x_1^{i_1}\dots x_n^{i_n}y_1^{j_1}\dots y_n^{j_n}w$ are linearly independent in H_c. To do this, it suffices to show that the images of these monomials under the homomorphism ϕ, i.e. $x_1^{i_1}\dots x_n^{i_n}D_{x_1}^{j_1}\dots D_{x_n}^{j_n}w$, are linearly independent.

Given an element $A\in\mathcal{A}$, writing $A=\sum_{w\in W}P_w w$ with $P_w\in\mathcal{D}(U)$ we define the order of A, ord$A$, as the maximum of the orders of the P_w's. Notice that ord$AB\leq$ord$A$+ord$B$. We now remark that for any sequence of non negative indices $(i_1,\dots,i_n)$,

$$D_{x_1}^{i_1}\cdots D_{x_n}^{i_n}=\partial_{x_1}^{i_1}\cdots\partial_{x_n}^{i_n}+l.o.t.$$

Indeed this is true for D_{x_i}. We proceed by induction on $r=i_1+\cdots+i_n$. We can clearly assume $i_1>0$, so by induction,

$$D_{x_1}^{i_1}\cdots D_{x_n}^{i_n}=(\partial_{x_1}+l.o.t.)(\partial_{x_1}^{i_1-1}\cdots\partial_{x_n}^{i_n}+l.o.t.)=\partial_{x_1}^{i_1}\cdots\partial_{x_n}^{i_n}+l.o.t.$$

From this we deduce that for any pair of multiindices $I=(i_1,\dots,i_n)$, $J=(j_1,\dots,j_n)$, $w\in W$, setting $x_I={x_1}^{i_1}\cdots{x_n}^{i_n}$, $D_J=D_{x_1}^{j_1}\cdots D_{x_n}^{j_n}$, $\partial_J=\partial_{x_1}^{j_1}\cdots\partial_{x_n}^{j_n}$, we have

$$x_ID_Jw=x_I\partial_Jw+l.o.t.$$

Using this and the linear independence of the elements $x_I\partial_Jw$, it is immediate to conclude that the elements x_ID_Jw are linearly independent, proving our claim. □

REMARK 1. We see that the homomorphism ϕ identifies H_c with the subalgebra of $\mathcal{A}$ generated by $\mathbf{C}[\mathfrak{h}]$, the Dunkl operators D_y, $y\in\mathfrak{h}$ and W.

REMARK 2. Another way to state the PBW theorem is the following. Let $F^\bullet$ be a filtration on H_c defined by $\deg(x_i)=\deg(y_i)=1$, $\deg(w)=0$. Then we have a natural surjective mapping from $\mathbf{C}[\mathfrak{h}\times\mathfrak{h}^*]\rtimes W$ to the associated graded algebra $\mathrm{gr}(H_c)$. The PBW theorem claims that this map is in fact an isomorphism.

3.5 The spherical subalgebra

Let us now introduce the idempotent

$$e=\frac{1}{W}\sum_{w\in W}w\in\mathbf{C}[W].$$