

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 49 (2003)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA  
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**Kapitel:** 3.5 The spherical subalgebra  
**DOI:** <https://doi.org/10.5169/seals-66677>

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*Proof.* It is easy to see that the map  $\mu$  is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials  $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} w$  are linearly independent in  $H_c$ . To do this, it suffices to show that the images of these monomials under the homomorphism  $\phi$ , i.e.  $x_1^{i_1} \dots x_n^{i_n} D_{x_1}^{j_1} \dots D_{x_n}^{j_n} w$ , are linearly independent.

Given an element  $A \in \mathcal{A}$ , writing  $A = \sum_{w \in W} P_w w$  with  $P_w \in \mathcal{D}(U)$  we define the order of  $A$ ,  $\text{ord}A$ , as the maximum of the orders of the  $P_w$ 's. Notice that  $\text{ord}AB \leq \text{ord}A + \text{ord}B$ . We now remark that for any sequence of non negative indices  $(i_1, \dots, i_n)$ ,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + l.o.t.$$

Indeed this is true for  $D_{x_i}$ . We proceed by induction on  $r = i_1 + \dots + i_n$ . We can clearly assume  $i_1 > 0$ , so by induction,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = (\partial_{x_1} + l.o.t.)(\partial_{x_1}^{i_1-1} \dots \partial_{x_n}^{i_n} + l.o.t.) = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + l.o.t.$$

From this we deduce that for any pair of multiindices  $I = (i_1, \dots, i_n)$ ,  $J = (j_1, \dots, j_n)$ ,  $w \in W$ , setting  $x_I = x_1^{i_1} \dots x_n^{i_n}$ ,  $D_J = D_{x_1}^{j_1} \dots D_{x_n}^{j_n}$ ,  $\partial_J = \partial_{x_1}^{j_1} \dots \partial_{x_n}^{j_n}$ , we have

$$x_I D_J w = x_I \partial_J w + l.o.t.$$

Using this and the linear independence of the elements  $x_I \partial_J w$ , it is immediate to conclude that the elements  $x_I D_J w$  are linearly independent, proving our claim.  $\square$

REMARK 1. We see that the homomorphism  $\phi$  identifies  $H_c$  with the subalgebra of  $\mathcal{A}$  generated by  $\mathbf{C}[\mathfrak{h}]$ , the Dunkl operators  $D_y$ ,  $y \in \mathfrak{h}$  and  $W$ .

REMARK 2. Another way to state the PBW theorem is the following. Let  $F^\bullet$  be a filtration on  $H_c$  defined by  $\deg(x_i) = \deg(y_i) = 1$ ,  $\deg(w) = 0$ . Then we have a natural surjective mapping from  $\mathbf{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$  to the associated graded algebra  $\text{gr}(H_c)$ . The PBW theorem claims that this map is in fact an isomorphism.

### 3.5 THE SPHERICAL SUBALGEBRA

Let us now introduce the idempotent

$$e = \frac{1}{W} \sum_{w \in W} w \in \mathbf{C}[W].$$

DEFINITION 3.13. The *spherical subalgebra* of  $H_c$  is the algebra  $eH_c e$ .

Notice that  $1 \notin eH_c e$ . On the other hand, since  $ex = xe = e$  for  $x \in eH_c e$ ,  $e$  is the unit for the spherical subalgebra. We can embed both  $\mathbf{C}[\mathfrak{h}^*]^W$  and  $\mathbf{C}[\mathfrak{h}]^W$  in the spherical subalgebra as follows. Take  $f \in \mathbf{C}[\mathfrak{h}^*]^W$  (the other case is identical) and set  $m_e(f) = fe$ . Since  $f$  is invariant, we have  $efe = fe^2 = fe = m_e(f)$ , so that  $m_e$  actually maps  $\mathbf{C}[\mathfrak{h}^*]^W$  to  $eH_c e$ . The injectivity is clear from the PBW-theorem. As for the fact that  $m_e$  is a homomorphism, we have  $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$ . From now on, we will consider both  $\mathbf{C}[\mathfrak{h}^*]^W$  and  $\mathbf{C}[\mathfrak{h}]^W$  as subalgebras of the spherical subalgebra.

### 3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras  $H_c$  and  $eH_c e$ .

DEFINITION 3.14. The category  $\mathcal{O}(H_c)$  (resp.  $\mathcal{O}(eH_c e)$ ) is the full subcategory of the category of  $H_c$ -modules (resp.  $eH_c e$ -modules) whose objects are the modules  $M$  such that

- 1)  $M$  is finitely generated.
- 2) For all  $v \in M$ , the subspace  $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$  is finite dimensional.

We can define a functor

$$F: \mathcal{O}(H_c) \rightarrow \mathcal{O}(eH_c e)$$

by setting  $F(M) = eM$ . It is easy to show that  $F(M)$  is an object of  $\mathcal{O}(eH_c e)$ .

We are now going to explain how to construct some modules in  $\mathcal{O}(H_c)$  which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take  $\lambda \in \mathfrak{h}^*$ . Denote by  $W_\lambda \subset W$  the stabilizer of  $\lambda$ . Take an irreducible  $W_\lambda$ -module  $\tau$ . We define a structure of  $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$ -module on  $\tau$  by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau, w \in W_\lambda, f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by  $\lambda \# \tau$ . We can then consider the  $H_c$ -module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]} \lambda \# \tau.$$

This is called a Whittaker module. In the special case  $\lambda = 0$  (and hence  $W_\lambda = W$ ), the module  $M(0, \tau)$  is called a Verma module. It is clear that these are objects of  $\mathcal{O}$ . Notice that as  $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$ -module,  $M(\lambda, \tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$ .