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Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA

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Kapitel: 3.5 The spherical subalgebra

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Proof. It is easy to see that the map μ is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials $x_1^{i_1}\dots x_n^{i_n}y_1^{j_1}\dots y_n^{j_n}w$ are linearly independent in H_c. To do this, it suffices to show that the images of these monomials under the homomorphism ϕ, i.e. $x_1^{i_1}\dots x_n^{i_n}D_{x_1}^{j_1}\dots D_{x_n}^{j_n}w$, are linearly independent.

Given an element $A\in\mathcal{A}$, writing $A=\sum_{w\in W}P_w w$ with $P_w\in\mathcal{D}(U)$ we define the order of A, ord$A$, as the maximum of the orders of the P_w's. Notice that ord$AB\leq$ord$A$+ord$B$. We now remark that for any sequence of non negative indices $(i_1,\dots,i_n)$,

$$D_{x_1}^{i_1}\cdots D_{x_n}^{i_n}=\partial_{x_1}^{i_1}\cdots\partial_{x_n}^{i_n}+l.o.t.$$

Indeed this is true for D_{x_i}. We proceed by induction on $r=i_1+\cdots+i_n$. We can clearly assume $i_1>0$, so by induction,

$$D_{x_1}^{i_1}\cdots D_{x_n}^{i_n}=(\partial_{x_1}+l.o.t.)(\partial_{x_1}^{i_1-1}\cdots\partial_{x_n}^{i_n}+l.o.t.)=\partial_{x_1}^{i_1}\cdots\partial_{x_n}^{i_n}+l.o.t.$$

From this we deduce that for any pair of multiindices $I=(i_1,\dots,i_n)$, $J=(j_1,\dots,j_n)$, $w\in W$, setting $x_I={x_1}^{i_1}\cdots{x_n}^{i_n}$, $D_J=D_{x_1}^{j_1}\cdots D_{x_n}^{j_n}$, $\partial_J=\partial_{x_1}^{j_1}\cdots\partial_{x_n}^{j_n}$, we have

$$x_ID_Jw=x_I\partial_Jw+l.o.t.$$

Using this and the linear independence of the elements $x_I\partial_Jw$, it is immediate to conclude that the elements x_ID_Jw are linearly independent, proving our claim. □

REMARK 1. We see that the homomorphism ϕ identifies H_c with the subalgebra of $\mathcal{A}$ generated by $\mathbf{C}[\mathfrak{h}]$, the Dunkl operators D_y, $y\in\mathfrak{h}$ and W.

REMARK 2. Another way to state the PBW theorem is the following. Let $F^\bullet$ be a filtration on H_c defined by $\deg(x_i)=\deg(y_i)=1$, $\deg(w)=0$. Then we have a natural surjective mapping from $\mathbf{C}[\mathfrak{h}\times\mathfrak{h}^*]\rtimes W$ to the associated graded algebra $\mathrm{gr}(H_c)$. The PBW theorem claims that this map is in fact an isomorphism.

3.5 THE SPHERICAL SUBALGEBRA

Let us now introduce the idempotent

$$e=\frac{1}{W}\sum_{w\in W}w\in\mathbf{C}[W].$$

DEFINITION 3.13. The *spherical subalgebra* of H_c is the algebra eH_ce.

Notice that $1 \notin eH_ce$. On the other hand, since $ex = xe = e$ for $x \in eH_ce$, e is the unit for the spherical subalgebra. We can embed both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ in the spherical subalgebra as follows. Take $f \in \mathbf{C}[\mathfrak{h}^*]^W$ (the other case is identical) and set $m_e(f) = fe$. Since f is invariant, we have $efe = fe^2 = fe = m_e(f)$, so that m_e actually maps $\mathbf{C}[\mathfrak{h}^*]^W$ to eH_ce. The injectivity is clear from the PBW-theorem. As for the fact that m_e is a homomorphism, we have $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$. From now on, we will consider both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ as subalgebras of the spherical subalgebra.

3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras H_c and eH_ce.

DEFINITION 3.14. The category $\mathcal{O}(H_c)$ (resp. $\mathcal{O}(eH_ce)$) is the full subcategory of the category of H_c-modules (resp. eH_ce-modules) whose objects are the modules M such that

1) M is finitely generated.
2) For all $v \in M$, the subspace $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$ is finite dimensional.

We can define a functor

$$F\colon \mathcal{O}(H_c) \to \mathcal{O}(eH_ce)$$

by setting $F(M) = eM$. It is easy to show that $F(M)$ is an object of $\mathcal{O}(eH_ce)$.

We are now going to explain how to construct some modules in $\mathcal{O}(H_c)$ which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take $\lambda \in \mathfrak{h}^*$. Denote by $W_\lambda \subset W$ the stabilizer of λ. Take an irreducible W_λ-module τ. We define a structure of $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$-module on τ by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau,\ w \in W_\lambda,\ f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by $\lambda\#\tau$. We can then consider the H_c-module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]} \lambda\#\tau .$$

This is called a Whittaker module. In the special case $\lambda = 0$ (and hence $W_\lambda = W$), the module $M(0, \tau)$ is called a Verma module. It is clear that these are objects of $\mathcal{O}$. Notice that as $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(\lambda, \tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$.