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DEFINITION 3.13. The *spherical subalgebra* of  $H_c$  is the algebra  $eH_c e$ .

Notice that  $1 \notin eH_c e$ . On the other hand, since  $ex = xe = e$  for  $x \in eH_c e$ ,  $e$  is the unit for the spherical subalgebra. We can embed both  $\mathbf{C}[\mathfrak{h}^*]^W$  and  $\mathbf{C}[\mathfrak{h}]^W$  in the spherical subalgebra as follows. Take  $f \in \mathbf{C}[\mathfrak{h}^*]^W$  (the other case is identical) and set  $m_e(f) = fe$ . Since  $f$  is invariant, we have  $efe = fe^2 = fe = m_e(f)$ , so that  $m_e$  actually maps  $\mathbf{C}[\mathfrak{h}^*]^W$  to  $eH_c e$ . The injectivity is clear from the PBW-theorem. As for the fact that  $m_e$  is a homomorphism, we have  $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$ . From now on, we will consider both  $\mathbf{C}[\mathfrak{h}^*]^W$  and  $\mathbf{C}[\mathfrak{h}]^W$  as subalgebras of the spherical subalgebra.

### 3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras  $H_c$  and  $eH_c e$ .

DEFINITION 3.14. The category  $\mathcal{O}(H_c)$  (resp.  $\mathcal{O}(eH_c e)$ ) is the full subcategory of the category of  $H_c$ -modules (resp.  $eH_c e$ -modules) whose objects are the modules  $M$  such that

- 1)  $M$  is finitely generated.
- 2) For all  $v \in M$ , the subspace  $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$  is finite dimensional.

We can define a functor

$$F: \mathcal{O}(H_c) \rightarrow \mathcal{O}(eH_c e)$$

by setting  $F(M) = eM$ . It is easy to show that  $F(M)$  is an object of  $\mathcal{O}(eH_c e)$ .

We are now going to explain how to construct some modules in  $\mathcal{O}(H_c)$  which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take  $\lambda \in \mathfrak{h}^*$ . Denote by  $W_\lambda \subset W$  the stabilizer of  $\lambda$ . Take an irreducible  $W_\lambda$ -module  $\tau$ . We define a structure of  $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$ -module on  $\tau$  by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau, w \in W_\lambda, f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by  $\lambda \# \tau$ . We can then consider the  $H_c$ -module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]} \lambda \# \tau.$$

This is called a Whittaker module. In the special case  $\lambda = 0$  (and hence  $W_\lambda = W$ ), the module  $M(0, \tau)$  is called a Verma module. It is clear that these are objects of  $\mathcal{O}$ . Notice that as  $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$ -module,  $M(\lambda, \tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$ .

EXAMPLE 3.15. If  $\lambda = 0$  and  $\tau = \mathbf{1}$  is the trivial representation of  $W$ , the Verma module  $M(0, \mathbf{1}) = \mathbf{C}[\mathfrak{h}]$ . The action of  $\mathbf{C}[\mathfrak{h}]$  is given by multiplication, that of  $\mathbf{C}[\mathfrak{h}^*]$  is generated by the Dunkl operators and  $W$  acts in the usual way.

### 3.7 GENERIC $c$

Opdam and Rouquier have recently studied the structure of the categories  $\mathcal{O}(H_c)$ ,  $\mathcal{O}(eH_c e)$ , and found that it is especially simple if  $c$  is “generic” in a certain sense. Namely, recall that for a  $W$ -invariant function  $q: \Sigma \rightarrow \mathbf{C}^*$  one can define the *Hecke algebra*  $\text{He}_q(W)$  to be the quotient of the group algebra of the fundamental group of  $U/W$  by the relations  $(T_s - 1)(T_s + q_s) = 0$ , where  $T_s$  is the image in  $U/W$  of a small half-circle around the hyperplane of  $s$  in the counterclockwise direction. It is well known that  $\text{He}_q(W)$  is an algebra of dimension  $|W|$ , which coincides with  $\mathbf{C}[W]$  if  $q = 1$ . It is also known that  $\text{He}_q(W)$  is semisimple (and isomorphic to  $\mathbf{C}[W]$  as an algebra) unless  $q_s$  belongs for some  $s$  to a finite set of roots of unity depending on  $W$  (see [Hu]).

DEFINITION 3.16. The function  $c$  is said to be *generic* if for  $q = e^{2\pi ic}$ , the Hecke algebra  $\text{He}_q(W)$  is semisimple.

In particular, any irrational  $c$  is generic, and (more important for us) an integer valued  $c$  is generic (since in this case  $q = 1$ ). We can now state the following central result:

THEOREM 3.17 (Opdam-Rouquier [OR]; see also [BEG] for an exposition). *If  $c$  is generic (in particular, if  $c$  takes non negative integer values), then the irreducible objects in  $\mathcal{O}$  are exactly the modules  $M(\lambda, \tau)$ . Moreover, the category  $\mathcal{O}$  is semisimple.*

We also have

THEOREM 3.18 ([OR]). *If  $c$  is generic then the functor  $F$  is an equivalence of categories.*

From Theorem 3.17 we can deduce

THEOREM 3.19 ([BEG]). *If  $c$  is generic, then  $H_c$  is a simple algebra.*

In the case  $c = 0$ , we get the simplicity of  $\mathbf{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathbf{C}[W]$ , which is well known.