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The following important theorem shows that this action extends to  $Q_m$ .

**THEOREM 3.22 ([BEG]).** *There exists a unique representation of the algebra  $eH_me$  on  $Q_m$  in which an element  $q \in \mathbf{C}[\mathfrak{h}]^W$  acts by multiplication and an element  $q \in \mathbf{C}[\mathfrak{h}^*]^W$  by  $L_q$ .*

*Proof.* Since by Proposition 3.5,  $L_q$  preserves  $Q_m$ , we get a uniquely defined representation of the subalgebra of  $eH_me$  generated by  $\mathbf{C}[\mathfrak{h}]^W$  and  $\mathbf{C}[\mathfrak{h}^*]^W$  on  $Q_m$ . The result now follows from Theorem 3.21.  $\square$

### 3.10 PROOF OF THEOREM 1.8

Finally we can prove Theorem 1.8.

To do this, observe that as an  $eH_me$ -module,  $Q_m$  is in the category  $\mathcal{O}(eH_me)$ , and  $\mathbf{C}[\mathfrak{h}^*]^W$  acts locally nilpotently in  $Q_m$  (by degree arguments). We can now apply Theorem 3.18 and Theorem 3.17 and deduce that  $Q_m$  is a direct sum of modules of the form  $eM(0, \tau)$ . As a  $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$ -module,  $M(0, \tau) = \mathbf{C}[\mathfrak{h}] \otimes \tau$ . On the other hand, by Chevalley's theorem, there is an isomorphism  $\mathbf{C}[\mathfrak{h}] \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \mathbf{C}[W]$ , commuting with the action of  $W$  and  $\mathbf{C}[\mathfrak{h}]^W$ . Thus we get an isomorphisms of  $\mathbf{C}[\mathfrak{h}]^W$ -modules

$$eM(0, \tau) \simeq (M(0, \tau))^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes (\mathbf{C}[W] \otimes \tau)^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \tau,$$

proving that  $eM(0, \tau)$  and hence  $Q_m$  is a free  $\mathbf{C}[\mathfrak{h}]^W$ -module.  $\square$

**EXAMPLE 3.23.** For  $W = \mathbf{Z}/2$  and  $\mathfrak{h} = \mathbf{C}$ , take the polynomials  $1, x^{2m+1}$ . Notice that  $L(1) = L(x^{2m+1}) = 0$  while  $s(1) = 1, s(x^{2m+1}) = -x^{2m+1}, s \in \mathbf{Z}/2$  being the element of order two. It follows that  $Q_m$  as a  $eH_me$ -module is the direct sum of  $\mathbf{C}[x^2] \oplus x^{2m+1}\mathbf{C}[x^2]$ . These modules are irreducible. Moreover,  $\mathbf{C}[x^2] \simeq eM(0, \mathbf{1}), x^{2m+1}\mathbf{C}[x^2] \simeq eM(0, \varepsilon), \varepsilon$  being the sign representation.

### 3.11 PROOF OF THEOREM 1.15

Let  $I$  be a nonzero two-sided ideal in  $\mathcal{D}(X_m)$ . First we claim that  $I$  nontrivially intersects  $Q_m$ . Indeed, otherwise let  $K \in I$  be a lowest order nonzero element in  $I$ . Since the order of  $K$  is positive, there exists  $f \in Q_m$  such that  $[K, f] \neq 0$ . Then  $[K, f] \in I$  is of smaller order than  $K$ , a contradiction.

Now let  $f \in Q_m$  be an element of  $I$ . Then  $g = \prod_{w \in W} {}^w f \in I$ . But  $g$  is  $W$ -invariant. This shows that the intersection  $J$  of  $I$  with the subalgebra  $H_m$  in  $\mathcal{D}(X_m)$  is nonzero. But  $H_m$  is simple by Theorem 3.19, so  $J = H_m$ . Hence,  $1 \in J \subset I$ , and  $I = \mathcal{D}(X_m)$ .  $\square$