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Autor: HÄMMERLI, Jean-François
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Now, as the principal subgroups are all conjugate by an element of G_o (see [8], Théorème, pp. 46–47), so are their centralizers. Therefore, the extensions $Z_o \hookrightarrow Z_G(H) \twoheadrightarrow \Gamma$, for H running through the family of principal subgroups, all belong to the same class. This shows that Θ is well defined and satisfies $\Theta \circ \Lambda = id_{\mathcal{E}(\Gamma, Z_o, \bar{\varphi})}$. As Λ is bijective, this shows that $\Theta = \Lambda^{-1}$. □

We summarize the situation exposed in this section.

THEOREM 3.3. *Suppose given* G_o, *a homomorphism* $\varphi \colon \Gamma \to \mathrm{Out}(G_o)$ *and an extension* $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$, *for which the homomorphism* $\Gamma \to \mathrm{Aut}(Z_o)$ *coincides with* $\bar{\varphi}$. *Then, up to equivalence of extensions, there exists a unique compact Lie group* G *fitting into the commutative diagram*

$$\begin{array}{ccccc}
Z_o & \hookrightarrow & Z & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \downarrow \varphi \\
\mathrm{Inn}(G_o) & \hookrightarrow & \mathrm{Aut}(G_o) & \twoheadrightarrow & \mathrm{Out}(G_o)
\end{array}$$

where the rows are group extensions. Moreover the given data allow the construction of an extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$, *in which the subgroup* Z *is the centralizer of a principal subgroup.*

Conversely, the class of the extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$ *in* $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ *can be recovered by taking the centralizer of any principal subgroup.*

4. PROOF OF THE MAIN THEOREM AND EXAMPLES

We are almost ready to show that the map described in the Introduction is an action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on the set

$$\mathcal{E} \approx \coprod_{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Out}(G_o))} H^2_{\bar{\varphi}}(\Gamma; Z_o)\,.$$

We first introduce some notation. For an element g in a group K, we will write c_g for conjugation by g, i.e. $c_g(x) = gxg^{-1}$, for all x in K. For $\alpha \in \mathrm{Out}(G_o)$, we will choose $\widetilde{\alpha} \in \mathrm{Aut}(G_o)$ such that $\pi(\widetilde{\alpha}) = \alpha$, and we will denote the restricted automorphism by $\bar{\alpha} \in \mathrm{Aut}(Z_o)$. Finally, recall that a cohomology class $u \in H^2_{\overline{\varphi}}(\Gamma; Z_o)$ is canonically identified with the corresponding equivalence class of extensions and is denoted by $u = \left[Z_o \overset{\mu}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma\right]$.

LEMMA 4.1.

(i) *The map* $\mathrm{Out}(G_o) \times \mathcal{E} \to \mathcal{E},\ (\alpha, u) \mapsto u\,.\,\alpha = \alpha^*(u) = \left[Z_o \overset{\mu\circ\bar{\alpha}}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma\right]$ *defines a right action. The image* $\alpha^*(u)$ *corresponds to the extension* $G_o \overset{i\circ\tilde{\alpha}}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} \Gamma$, *and belongs to* $H^2_{\overline{\psi}}(\Gamma; Z_o) \subset \mathcal{E}$, *where* $\psi = c_{\alpha^{-1}} \circ \varphi$.

(ii) *The map* $\mathrm{Aut}(\Gamma) \times \mathcal{E} \to \mathcal{E},\ (\beta, u) \mapsto \beta\,.\,u = \beta_*(u) = \left[Z_o \overset{\mu}{\hookrightarrow} Z \overset{\beta\circ\nu}{\twoheadrightarrow} \Gamma\right]$ *defines a left action. The image* $\beta_*(u)$ *corresponds to the extension* $G_o \overset{i}{\hookrightarrow} G \overset{\beta\circ p}{\twoheadrightarrow} \Gamma$, *and belongs to* $H^2_{\overline{\theta}}(\Gamma; Z_o) \subset \mathcal{E}$, *where* $\theta = \varphi \circ \beta^{-1}$.

Proof. As the proofs of the two parts of the lemma are very similar, we only treat the first one. Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
Z_o & \xrightarrow{\bar{\alpha}} & Z_o & \overset{\mu}{\hookrightarrow} & Z = Z_G(H) & \overset{\nu}{\twoheadrightarrow} & \Gamma \\
\downarrow & & \downarrow & & \downarrow & & \| \\
G_o & \xrightarrow{\tilde{\alpha}} & G_o & \overset{i}{\hookrightarrow} & G & \overset{p}{\twoheadrightarrow} & \Gamma \\
\downarrow & & & & \downarrow c & & \downarrow \psi \\
\mathrm{Inn}(G_o) & & \hookrightarrow & & \mathrm{Aut}(G_o) & \overset{\pi}{\twoheadrightarrow} & \mathrm{Out}(G_o)
\end{array}
$$

The principal subgroups are preserved by isomorphisms. As $\tilde{\alpha}^{-1}(H)$ is clearly centralized by any element in Z, the statement about which extension corresponds to $\alpha^*(u)$ follows from Theorem 3.3. At the same time, this shows that the map is well defined. It is then straightforward to check that it is a right action. For the resulting homomorphism, we choose a set theoretic section $v\colon \Gamma \to G$ of $p\colon G \twoheadrightarrow \Gamma$, and compute for $\gamma \in \Gamma$:

$$
\begin{aligned}
\psi(\gamma) &= \pi\big((i \circ \tilde{\alpha})^{-1} \circ c_{v(\gamma)} \circ (i \circ \tilde{\alpha})\big) \\
&= \pi\big(\tilde{\alpha}^{-1} \circ (i^{-1} \circ c_{v(\gamma)} \circ i) \circ \tilde{\alpha}\big) \\
&= \pi(\tilde{\alpha})^{-1} \circ \pi(i^{-1} \circ c_{v(\gamma)} \circ i) \circ \pi(\tilde{\alpha}) \\
&= \alpha^{-1} \circ \varphi(\gamma) \circ \alpha \\
&= (c_{\alpha^{-1}} \circ \varphi)(\gamma)\,.
\end{aligned}
$$

□

Clearly for $u \in \mathcal{E}$ and a corresponding representative $G_o \hookrightarrow G_u \twoheadrightarrow \Gamma$, we have $G_u \cong G_{\alpha^*(u)} \cong G_{\beta_*(u)}$ for all $\alpha \in \mathrm{Out}(G_o)$, $\beta \in \mathrm{Aut}(\Gamma)$. Moreover, it is clear that the two actions commute and so we get a left action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on $\mathcal{E}$. Elements in the same orbit represent isomorphic groups; the main result of this paper, stated in the Introduction, tells that the converse is true.

Proof of the Main Theorem: Let $\rho \colon G_{u_1} \to G_{u_2}$ be an isomorphism of compact Lie groups. As the connected component of the identity is preserved by an isomorphism, this gives rise to the commutative diagram

$$\begin{array}{ccccc} G_o & \hookrightarrow & G_{u_1} & \twoheadrightarrow & \Gamma \\ \cong \downarrow \tilde{\rho} & & \cong \downarrow \rho & & \cong \downarrow \beta \\ G_o & \hookrightarrow & G_{u_2} & \twoheadrightarrow & \Gamma \end{array}$$

Let us define $\alpha = \pi(\tilde{\rho}) \in \mathrm{Out}(G_o)$ and $\bar{\alpha} = \rho|_{Z_o}$. As the centralizers of principal subgroups are preserved by isomorphisms, and by Theorem 3.3, this induces a new commutative diagram that we write as follows:

$$\begin{array}{ccccc} Z_o & \xrightarrow{\mu_1 \circ \bar{\alpha}^{-1}} & Z_{u_1} & \xrightarrow{\beta \circ \nu_1} & \Gamma \\ \| & & \cong \downarrow \bar{\rho} = \rho|_{Z_{u_1}} & & \| \\ Z_o & \xrightarrow[\mu_2]{} & Z_{u_2} & \xrightarrow[\nu_2]{} & \Gamma \end{array}$$

Thus, by Lemma 4.1, we have $u_2 = (\alpha^{-1})^* \beta_*(u_1)$, and so u_1 and u_2 are in the same orbit. □

REMARK 4.2. The extension $\mathrm{Inn}(G_o) \overset{\iota}{\hookrightarrow} \mathrm{Aut}(G_o) \overset{\pi}{\twoheadrightarrow} \mathrm{Out}(G_o)$ is split; however, other facts are relevant for allowing in the Main Theorem the passage from *up to equivalence* to *up to isomorphism*. The crucial point is that the class of extensions of the center of the connected component of the identity G_o can be represented by subgroups of G, namely centralizers of principal subgroups, that are preserved by isomorphisms and all conjugate by elements in G_o. This also raises two natural questions: are there larger classes of groups for which the Main Theorem holds, and also, can one find explicit examples for which it fails (even when supposing that the extension relating the automorphism groups of the kernel of the extension is split)?

Before proceeding with two examples, we introduce notations for three elements of the group SU(2), which will also appear in the final proposition of the paper. We denote the identity matrix by $\mathbf{1}$ and we set $-\mathbf{1} = \mathrm{diag}(-1,-1)$. We also set

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

EXAMPLE 4.3. We take $G_o = \mathrm{SU}(2)$ and $\Gamma = \mathbf{Z}/2$. As $\mathrm{Out}(G_o)$ is trivial and $Z_o \cong \mathbf{Z}/2$, we have

$$\mathcal{E}(\mathbf{Z}/2, \mathrm{SU}(2)) \approx \coprod_{\varphi \in \mathrm{Hom}(\mathbf{Z}/2,0)} H^2_{\overline{\varphi}}(\mathbf{Z}/2;\mathbf{Z}/2) = H^2(\mathbf{Z}/2;\mathbf{Z}/2) \cong \mathbf{Z}/2 .$$

The group $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ being trivial, these two elements correspond to two non-isomorphic compact Lie groups. The first one is clearly $G_{u_0} = \mathrm{SU}(2) \times \mathbf{Z}/2$. Let us give a description of the second one. Conjugating a matrix in SU(2) by j amounts to taking the complex conjugate of each entry in the matrix, i.e. $c_j\colon \mathrm{SU}(2) \to \mathrm{SU}(2)$, $g \mapsto c_j(g) = \bar{g}$. Let us denote by $G_{u_1} = \mathrm{SU}(2) \rtimes_j \mathbf{Z}/2$ the semidirect product where the generator t of $\mathbf{Z}/2$ acts as c_j on SU(2). As the center of G_{u_1} is given by $\langle (j,t) \rangle \cong \mathbf{Z}/4$, G_{u_0} and G_{u_1} are non-isomorphic. Therefore G_{u_1} is the second compact Lie group that we were looking for.

It is clear, from what has been done so far, that the elements in $H^2_{\overline{\varphi}}(\Gamma; Z_o)$ and in $H^2_{\overline{\psi}}(\Gamma; Z_o)$, with $\psi = c_{\alpha^{-1}} \circ \varphi$, will be identified (at least) pairwise under the action of the element $\alpha \in \mathrm{Out}(G_o)$. The second example is intended to show that identifications can even occur inside a given cohomology group (i.e. without changing the "outer" action of Γ on G_o).

EXAMPLE 4.4. We take $G_o = \mathrm{SU}(2)\times\mathrm{SU}(2) \cong \mathrm{Spin}(4)$ and keep $\Gamma = \mathbf{Z}/2$. The outer automorphism group is given by $\mathrm{Out}(G_o) = \langle \tau \rangle$, where $\widetilde{\tau}$ is the automorphism that exchanges the two factors, i.e.

$$\widetilde{\tau}\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2),\ (g,h) \longmapsto (h,g),$$

and $Z_o = \mathbf{Z}/2 \times \mathbf{Z}/2$. We thus have

$$\begin{aligned}\mathcal{E}(\mathbf{Z}/2, \mathrm{SU}(2) \times \mathrm{SU}(2)) &\approx \coprod_{\varphi \in \mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}/2)} H^2_{\overline{\varphi}}(\mathbf{Z}/2;\mathbf{Z}/2 \times \mathbf{Z}/2) \\ &= H^2(\mathbf{Z}/2;\mathbf{Z}/2 \times \mathbf{Z}/2) \amalg H^2_{\overline{id}}(\mathbf{Z}/2;\mathbf{Z}/2 \times \mathbf{Z}/2) \\ &\approx (\mathbf{Z}/2 \times \mathbf{Z}/2) \amalg \{0\} .\end{aligned}$$

One then verifies that as extensions of the center, i.e. as centralizers of a principal subgroup, these five non-equivalent extensions are in fact represented by only three non-isomorphic groups, namely

$$\begin{aligned} Z_{u_0} &\cong \mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2 \\ Z_{u_1} &\cong \mathbf{Z}/2 \times \mathbf{Z}/4 \\ Z_{u_2} &\cong \mathbf{Z}/2 \times \mathbf{Z}/4 \\ Z_{u_3} &\cong \mathbf{Z}/2 \times \mathbf{Z}/4 \end{aligned}$$

for the elements of $H^2(\mathbf{Z}/2; \mathbf{Z}/2 \times \mathbf{Z}/2))$, and

$$Z_{v_0} \cong (\mathbf{Z}/2 \times \mathbf{Z}/2) \rtimes \mathbf{Z}/2 \cong \mathrm{D}_8$$

(where D_8 denotes the group of symmetries of the square) for the element of $H^2_{\overline{id}}(\mathbf{Z}/2; \mathbf{Z}/2 \times \mathbf{Z}/2)$. The group $\mathbf{Z}/2 \times \mathbf{Z}/4$ yields three non-equivalent extensions, because among its three elements of order 2, only one is divisible by 2 (the element $(0,2)$ in additive notation). Therefore, this element must be characteristic and changing the non-trivial element of $\mathbf{Z}/2 \times \mathbf{Z}/2$ that is mapped to it gives three extensions that must clearly be non-equivalent. At the level of Lie groups, the five non-equivalent extensions are represented by

$$\begin{aligned} G_{u_0} &= \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbf{Z}/2 \\ G_{u_1} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{j \times id} \mathbf{Z}/2 \\ G_{u_2} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{id \times j} \mathbf{Z}/2 \\ G_{u_3} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{j \times j} \mathbf{Z}/2 \\ G_{v_0} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{\tau} \mathbf{Z}/2 . \end{aligned}$$

(One checks that $(-\mathbf{1}, \mathbf{1}, e)$ corresponds to the characteristic element of order 2 in Z_{u_1} whereas it is $(\mathbf{1}, -\mathbf{1}, e)$ in Z_{u_2}, and therefore G_{u_1} and G_{u_2} are certainly not equivalent.) Finally, the group $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma) \cong \mathbf{Z}/2$ acts on this set of equivalent extensions, and it is clear that the only non-trivial orbit is $\{G_{u_1}, G_{u_2}\}$. Therefore there are four non-isomorphic extensions of $\mathbf{Z}/2$ by $\mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{Spin}(4)$.