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ATIYAH'S L^2 -INDEX THEOREM

by Indira CHATTERJI and Guido MISLIN

1. INTRODUCTION

The L^2 -Index Theorem of Atiyah [1] expresses the index of an elliptic operator on a closed manifold M in terms of the G -equivariant index of some regular covering \tilde{M} of M , with G the group of covering transformations. Atiyah's proof is analytic in nature. Our proof is algebraic and involves an embedding of a given group into an acyclic one, together with naturality properties of the indices.

2. REVIEW OF THE L^2 -INDEX THEOREM

The main reference for this section is Atiyah's paper [1]. All manifolds considered are smooth Riemannian, without boundary. Covering spaces of manifolds carry the induced smooth and Riemannian structure. Let M be a closed manifold and let E, F denote two complex (Hermitian) vector bundles over M . Consider an elliptic pseudo-differential operator

$$D: C^\infty(M, E) \rightarrow C^\infty(M, F)$$

acting on the smooth sections of the vector bundles. One defines its space of solutions

$$S_D = \{s \in C^\infty(M, E) \mid Ds = 0\} .$$

The complex vector space S_D has finite dimension (see [13]), and so has S_{D^*} the space of solutions of the adjoint D^* of D where

$$D^*: C^\infty(M, F) \rightarrow C^\infty(M, E)$$

is the unique continuous linear map satisfying

$$\langle Ds, s' \rangle = \int_M \langle Ds(m), s'(m) \rangle_F dm = \langle s, D^* s' \rangle = \int_M \langle s(m), D^* s'(m) \rangle_E dm$$

for all $s \in C^\infty(M, E)$, $s' \in C^\infty(M, F)$. One now defines the *index* of D as follows:

$$\text{Index}(D) = \dim_{\mathbf{C}}(S_D) - \dim_{\mathbf{C}}(S_{D^*}) \in \mathbf{Z}.$$

An explicit formula for $\text{Index}(D)$ is given by the famous Atiyah-Singer Theorem (cf. [2]). Consider a not necessarily connected, regular covering $\pi: \tilde{M} \rightarrow M$ with countable covering transformation group G . The projection π can be used to define an elliptic operator

$$\tilde{D} := \pi^*(D): C_c^\infty(\tilde{M}, \pi^*E) \rightarrow C_c^\infty(\tilde{M}, \pi^*F).$$

Denote by $S_{\tilde{D}}$ the closure of $\{s \in C_c^\infty(\tilde{M}, \pi^*E) \mid \tilde{D}s = 0\}$ in $L^2(\tilde{M}, \pi^*E)$. Let \tilde{D}^* denote the adjoint of \tilde{D} . The space $S_{\tilde{D}}$ is not necessarily finite dimensional, but being a closed G -invariant subspace of the L^2 -completion $L^2(\tilde{M}, \pi^*E)$ of the space of smooth sections with compact supports $C_c^\infty(\tilde{M}, \pi^*E)$, its von Neumann dimension is therefore defined as follows. Write

$$\mathcal{N}(G) = \{P: \ell^2(G) \rightarrow \ell^2(G) \text{ bounded and } G\text{-invariant}\}$$

for the group von Neumann algebra of G , where G acts on $\ell^2(G)$ via the right regular representation. Then $S_{\tilde{D}}$ is a finitely generated Hilbert G -module and hence can be represented by an idempotent matrix $P = (p_{ij}) \in M_n(\mathcal{N}(G))$ (recall that a finitely generated Hilbert G -module is isometrically G -isomorphic to a Hilbert G -subspace of the Hilbert space $\ell^2(G)^n$ for some $n \geq 1$, see [9]). One then sets

$$\dim_G(S_{\tilde{D}}) = \sum_{i=1}^n \langle p_{ii}(e), e \rangle = \kappa(P) \in \mathbf{R},$$

where by abuse of notation e denotes the element in $\ell^2(G)$ taking value 1 on the neutral element $e \in G$ and 0 elsewhere (see Eckmann's survey [9] on L^2 -cohomology for more on von Neumann dimensions). The map $\kappa: M_n(\mathcal{N}(G)) \rightarrow \mathbf{C}$ is the Kaplansky trace. One defines the L^2 -index of \tilde{D} by

$$\text{Index}_G(\tilde{D}) = \dim_G(S_{\tilde{D}}) - \dim_G(S_{\tilde{D}^*}).$$

We can now state Atiyah's L^2 -Index Theorem.

THEOREM 2.1 (Atiyah [1]). *For D an elliptic pseudo-differential operator on a closed Riemannian manifold M*

$$\text{Index}(D) = \text{Index}_G(\tilde{D})$$

for any countable group G and any lift \tilde{D} of D to a regular G -cover \tilde{M} of M .

In particular, the L^2 -index of \tilde{D} is always an integer, even though it is a priori given in terms of real numbers. The following serves as an illustration of the L^2 -Index Theorem.

EXAMPLE 2.2 (Atiyah's formula [1]). Let Ω^\bullet be the de Rham complex of complex valued differential forms on the closed connected manifold M and consider the de Rham differential $D = d + d^* : \Omega^{ev} \rightarrow \Omega^{odd}$. Let $\pi : \tilde{M} \rightarrow M$ be the universal cover of M so that $G = \pi_1(M)$. Then

- $\text{Index}(D) = \chi(M)$, the ordinary Euler characteristic of M .
- $\text{Index}_G(\tilde{D}) = \sum_j (-1)^j \beta^j(M)$, the L^2 -Euler characteristic of M .

The $\beta^j(M)$'s denote the L^2 -Betti numbers of M . Thus the L^2 -Index Theorem translates into Atiyah's formula

$$\chi(M) = \sum_j (-1)^j \beta^j(M).$$

We recall that the L^2 -Betti numbers $\beta^j(M)$ are in general not integers. For instance, if $\pi_1(M)$ is a finite group, one checks that

$$\beta^j(M) = \frac{1}{|\pi_1(M)|} b^j(\tilde{M}),$$

where $b^j(\tilde{M})$ stands for the ordinary j 'th Betti number of the universal cover \tilde{M} of M . In particular, for $1 < |\pi_1(M)| < \infty$, $\beta^0(M) = 1/|\pi_1(M)|$ is not an integer and the L^2 -Index Theorem reduces to the well-known fact that

$$\chi(M) = \frac{\chi(\tilde{M})}{|\pi_1(M)|}.$$

It is a conjecture (Atiyah Conjecture) that for a general closed connected manifold M the L^2 -Betti numbers $\beta^j(M)$ are always rational numbers, and even integers in case that $\pi_1(M)$ is torsion-free. For some interesting examples, which might lead to counterexamples, see Dicks and Schick [8].