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2. TILING AND INTEGER PROGRAMMING

Here we translate a polyomino tiling problem into an algebra question. Consider, for example, the problem of tiling the fairly simple shape

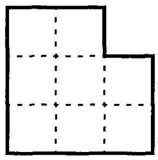


FIGURE 2.1
Region to tile with dominoes

by dominoes. For each possible tile placement, we introduce a variable, x_i, which indicates how many times that placement occurs in the tiling.

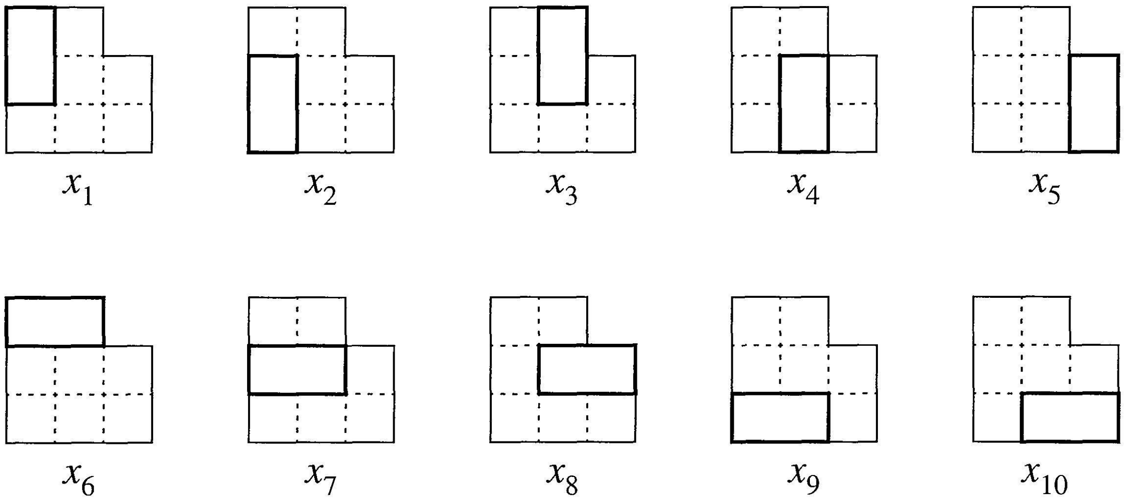


FIGURE 2.2
Possible tile placements and associated variables

In particular, its value will be either 0 or 1. Each cell of the region gives a linear equation, which indicates that the cell is covered exactly once. Thus, for the example of Figure 2.1, we get the system of linear equations

$$
\begin{array}{llllllllllll}
x_1 & & & & & +x_6 & & & & & = 1 \\
& & x_3 & & & +x_6 & & & & & = 1 \\
x_1 & +x_2 & & & & & +x_7 & & & & = 1 \\
& & x_3 & +x_4 & & & +x_7 & +x_8 & & & = 1 \\
& & & & x_5 & & & +x_8 & & & = 1 \\
& x_2 & & & & & & & +x_9 & & = 1 \\
& & & x_4 & & & & & +x_9 & +x_{10} & = 1 \\
& & & & x_5 & & & & & +x_{10} & = 1
\end{array}
\tag{2.3}
$$

A tiling then corresponds to a solution to the system above. However, the converse is not true; as noted above, the value of each variable must be either 0 or 1. A solution to the system in which every variable takes the value 0 or 1 indeed corresponds to a tiling.

Instead of making this requirement on the variables, it is sufficient (and perhaps more natural) to insist only that the values be non-negative integers. A linear system, such as (2.3) above, in which the coefficients are non-negative integers, where we seek solutions in non-negative integers, is one form of the *integer programming problem.* It is known that the general integer programming problem is NP-complete, see [16]. It has also been shown that the general problem of tiling a finite region by a set of polyominoes is NP-complete, see [6], [12].

LINEAR ALGEBRA AND SIGNED TILINGS

If we relax the condition that the variables take non-negative values, we have a more tractable, although somewhat different problem. Indeed, it is simply a linear algebra problem, albeit over $\mathbf{Z}$, but its resolution by row-reduction is straightforward.

A solution to (2.3) in integers, possibly negative, corresponds to a "signed tiling", i.e. where tiles may be subtracted from the region. Equivalently, we can think of allowing "anti-tiles". Again however, we do not quite have a one-to-one correspondence, because a signed tiling may utilize cells outside the region. Thus it is appropriate to consider all the cells of the square lattice when considering signed tilings.

TILE HOMOLOGY GROUPS

Following Conway and Lagarias, we define the tile homology group of a protoset $\mathcal{T}$. Let A be the free abelian group on all the cells of the square lattice. To a placement of a tile in $\mathcal{T}$, we associate the element of A which is 1 in those coordinates whose cell is covered by the tile placement, and is 0 in all other coordinates. Note that this element depends upon the particular placement of the tile. In the same way, to a region, we also associate an element of A. Again, this element depends upon the location and orientation of the region. For simplicity, we will consider a region to be a *fixed* subset of the square lattice.

DEFINITION 2.4. The *tile homology group* of $\mathcal{T}$ is the quotient $H(\mathcal{T}) = A/B(\mathcal{T})$, where $B(\mathcal{T}) \subseteq A$ is the subgroup generated by all elements corresponding to possible placements of tiles in $\mathcal{T}$.

The relevance of the tile homology group is clear. A region R has a tiling by $\mathcal{T}$ if and only if the element corresponding to R is in the submonoid of A generated by elements corresponding to tile placements, and it has a signed tiling if and only if the corresponding element is in $B(\mathcal{T})$. Thus $H(\mathcal{T})$ measures the obstruction to having a signed tiling by $\mathcal{T}$.

We introduce some conventions that will be useful. The cell with lower left corner at the point (i,j) we be called simply the (i,j) cell. We let a_{ij} denote the element of A corresponding to this cell, and let $\bar{a}_{ij}$ denote its image in $H(\mathcal{T})$.

The tile homology group is defined by infinitely many generators and infinitely many relations. In this form, it is somewhat difficult to use. In a number of simple cases, we can show that it is finitely generated.

EXAMPLE 2.5. $\mathcal{T} = \{$$\}$, both orientations allowed. $H(\mathcal{T})$ is defined by

Generators:	$\bar{a}_{ij}$	$i,j \in \mathbf{Z}$
Relations:	$\bar{a}_{ij} + \bar{a}_{i+1,j} = 0$	$i,j \in \mathbf{Z}$
	$\bar{a}_{ij} + \bar{a}_{i,j+1} = 0$	$i,j \in \mathbf{Z}$

Note that we have

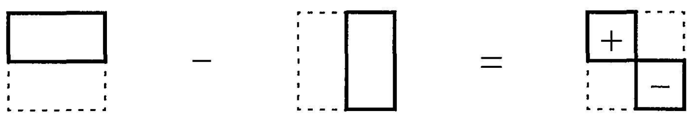
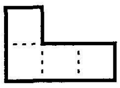


FIGURE 2.6
Translation of a square by 1 diagonal unit

which shows that $\bar{a}_{ij} - \bar{a}_{i+1,j-1} = 0$ in $H(\mathcal{T})$. Similarly, by rotating this figure by 90 degrees, we obtain $\bar{a}_{ij} - \bar{a}_{i+1,j+1} = 0$. These show that $H(\mathcal{T})$ is generated by the two elements $\bar{a}_{00}$ and $\bar{a}_{01}$. Now the relations above collapse into a single relation between these two generators: $\bar{a}_{00} + \bar{a}_{01} = 0$. Thus we see that $H(\mathcal{T}) \cong \mathbf{Z}$, and a specific isomorphism is given by $[R] \mapsto (b - r)$, where the region R has b black squares and r red squares. This shows that a region has a signed tiling by dominoes if and only if it has the same number of black squares as it has red squares.

EXAMPLE 2.7. $\mathcal{T} = \{$$\}$, all rotations and reflections allowed. From the equation

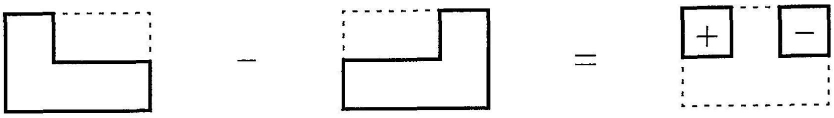


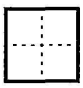
FIGURE 2.8
Translation of a square by 2 units

we see that $\bar{a}_{ij} = \bar{a}_{i+2,j}$, and similarly, we have $\bar{a}_{ij} = \bar{a}_{i,j+2}$. Thus $H(\mathcal{T})$ is generated by $\bar{a}_{00}, \bar{a}_{01}, \bar{a}_{10}$ and $\bar{a}_{11}$. The relations become

$$\begin{aligned}
2\bar{a}_{00} + \bar{a}_{01} + \bar{a}_{10} \phantom{+ \bar{a}_{11}} &= 0 \\
\bar{a}_{00} + 2\bar{a}_{01} \phantom{+ \bar{a}_{10}} + \bar{a}_{11} &= 0 \\
\bar{a}_{00} \phantom{+ \bar{a}_{01}} + 2\bar{a}_{10} + \bar{a}_{11} &= 0 \\
\bar{a}_{01} + \bar{a}_{10} + 2\bar{a}_{11} &= 0
\end{aligned}$$

so we easily find that $H(\mathcal{T}) \cong \mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$. A specific isomorphism is given by $[R] \mapsto (A - B - C + D, (2A + B - C) \bmod 4)$, where the region R contains A [respectively, B, C, D] (i,j) cells with i and j both even [respectively, i even and j odd, i odd and j even, i and j both odd]. From this analysis, we can easily find the numbering used in Figure 1.3 above.

In general, the tile homology group will not be finitely generated. A simple example that illustrates this is the following.

EXAMPLE 2.9. Let $\mathcal{T} = \{$$\}$. It is easy to show that $H(\mathcal{T})$ is a free abelian group on the generators which are images of all (i,j) cells with $i = 0$ or $j = 0$. In particular, $H(\mathcal{T})$ is not finitely generated.

We now show that the types of proofs given in the examples of the introduction can always be given using a suitable numbering of the cells of the square lattice.

PROPOSITION 2.10. *Let R be a region that does not have a signed tiling by the protoset $\mathcal{T}$. Then there is a numbering of all the cells in the square lattice with rational numbers such that*

(1) *any placement of a tile covers a total that is an integer, and*
(2) *the total covered by the region is not an integer.*

Proof. Let $r \in H(\mathcal{T})$ be the image of the region R in the tile homology group, which by hypothesis, is non-trivial. Let $\langle r \rangle \subseteq H(\mathcal{T})$ be the cyclic subgroup generated by r. Note that there is a homomorphism $\varphi\colon \langle r \rangle \to \mathbf{Q}/\mathbf{Z}$ with $\varphi(r) \neq 0$. For example, if r has infinite order, then φ may be defined by $\varphi(r) = \frac{1}{2} \bmod \mathbf{Z}$, while if r has finite order, $n > 1$, then we may take $\varphi(r) = \frac{1}{n} \bmod \mathbf{Z}$. Now $\mathbf{Q}/\mathbf{Z}$ is a divisible abelian group, so the homomorphism φ extends to a homomorphism $H(\mathcal{T}) \to \mathbf{Q}/\mathbf{Z}$, also called φ, which is defined on all of $H(\mathcal{T})$. Since A is a free abelian group, the composite map $A \twoheadrightarrow A/B(\mathcal{T}) = H(\mathcal{T}) \xrightarrow{\varphi} \mathbf{Q}/\mathbf{Z}$ lifts to a homomorphism $\psi\colon A \to \mathbf{Q}$, such that the following square commutes

$$\begin{array}{ccc} A & \xrightarrow{\psi} & \mathbf{Q} \\ \downarrow & & \downarrow \\ H(\mathcal{T}) & \xrightarrow{\varphi} & \mathbf{Q}/\mathbf{Z} \end{array}$$

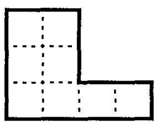
where the vertical arrows are the natural projections. Then ψ defines a numbering of the squares with rational numbers. Moreover, $B(\mathcal{T})$ is in the kernel of $A \to \mathbf{Q}/\mathbf{Z}$, which means that every tile placement covers an integral total. Also, R covers a total that is not an integer, because $\varphi(r) \neq 0$. □

REMARK 2.11. In many cases that we have examined, $H(\mathcal{T})$ is finitely generated, so that $\varphi(H(\mathcal{T})) \subseteq \mathbf{Q}/\mathbf{Z}$ is also finitely generated, whence $\varphi(H(\mathcal{T})) \subseteq \frac{1}{N}\mathbf{Z}/\mathbf{Z}$ for some integer N. In such cases, it seems convenient to clear denominators by multiplying everything by N. We thus obtain a numbering of the squares by integers, such that

($1'$) any placement of a tile covers a total divisible by N, and

($2'$) the region covers a total that is not divisible by N.

This shows that generalized checkerboard coloring arguments such as in [10, Thm. 6] can be given in a simpler form. We provide a numbering proof of Klarner's result, which is based upon his coloring.

PROPOSITION 2.12. *Let* $\mathcal{T} = \{$$\}$, *with all orientations allowed. If* $\mathcal{T}$ *tiles a rectangle, then its area is divisible by* 16.

Proof. We must show that $\mathcal{T}$ cannot tile a $(2m+1) \times (16n+8)$ rectangle or a $(4m+2) \times (8n+4)$ rectangle. Number the squares by

$$(i,j) \mapsto \begin{cases} 5 & \text{if } i \equiv 0 \bmod 4, \\ -3 & \text{if } i \equiv 2 \bmod 4, \text{ and} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

Then each tile covers a total of either 0 or 16, depending on its placement. In particular, it always covers a multiple of 16. However, a $(2m+1)\times(16n+8)$ rectangle covers a total that is congruent to 8 modulo 16, and so does a $(4m+2)\times(8n+4)$ rectangle. □

REMARK 2.13. Proposition 2.12 uses a single numbering to show that both types of rectangles cannot be tiled. In general, one may need several different numberings to show that several regions cannot be tiled.

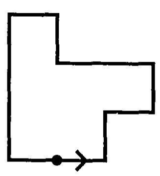
REMARK 2.14. It is not hard to show that we can translate a square by 4 units, and then it is straightforward to calculate that $H(\mathcal{T}) \cong \mathbf{Z}^5 \times (\mathbf{Z}/4\mathbf{Z})$.

3. BOUNDARY WORDS

In this section, we describe the boundary word method of Conway and Lagarias. This is a non-abelian analogue of tile homology, although that may not be immediately clear from the construction!

We must make an important assumption here. Our prototiles must be simply connected. We also assume that they have connected interior, although this condition can be relaxed in some cases. Such a tile has a *boundary word*, obtained by starting at a lattice point on the boundary, and traversing the boundary. For definiteness, we will always traverse in the counterclockwise direction. A unit step in the positive x [respectively, y] direction is transcribed as an x [respectively, y]. A step in the negative x [respectively, y] direction is transcribed as x^{-1} [respectively, y^{-1}].

EXAMPLE 3.1. Consider the following hexomino with the indicated base point.



$$x\,y\,x\,y\,x^{-2}y\,x^{-1}y^{-3}x$$

FIGURE 3.2

Example of boundary word