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$$(i,j) \mapsto \begin{cases} 5 & \text{if } i \equiv 0 \bmod 4, \\ -3 & \text{if } i \equiv 2 \bmod 4, \text{ and} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

Then each tile covers a total of either 0 or 16, depending on its placement. In particular, it always covers a multiple of 16. However, a $(2m+1) \times (16n+8)$ rectangle covers a total that is congruent to 8 modulo 16, and so does a $(4m+2) \times (8n+4)$ rectangle. $\square$

REMARK 2.13. Proposition 2.12 uses a single numbering to show that both types of rectangles cannot be tiled. In general, one may need several different numberings to show that several regions cannot be tiled.

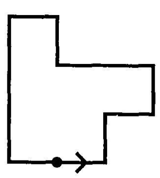
REMARK 2.14. It is not hard to show that we can translate a square by 4 units, and then it is straightforward to calculate that $H(\mathcal{T}) \cong \mathbf{Z}^5 \times (\mathbf{Z}/4\mathbf{Z})$.

3. BOUNDARY WORDS

In this section, we describe the boundary word method of Conway and Lagarias. This is a non-abelian analogue of tile homology, although that may not be immediately clear from the construction!

We must make an important assumption here. Our prototiles must be simply connected. We also assume that they have connected interior, although this condition can be relaxed in some cases. Such a tile has a *boundary word*, obtained by starting at a lattice point on the boundary, and traversing the boundary. For definiteness, we will always traverse in the counterclockwise direction. A unit step in the positive x [respectively, y] direction is transcribed as an x [respectively, y]. A step in the negative x [respectively, y] direction is transcribed as x^{-1} [respectively, y^{-1}].

EXAMPLE 3.1. Consider the following hexomino with the indicated base point.



$$x\,y\,x\,y\,x^{-2}y\,x^{-1}y^{-3}x$$

FIGURE 3.2

Example of boundary word

Its boundary word is $xyxyx^{-2}yx^{-1}y^{-3}x$. We note that the boundary word depends upon

(1) the choice of base point, and

(2) the particular orientation of the tile.

With regard to (1), a different base point gives rise to a conjugate boundary word. Condition (2) forces us to use translation-only tiles; therefore if we want to allow rotations and/or reflections, we must explicitly include each valid orientation in our protoset. This is actually advantageous, because we may use this to restrict the orientations that occur, for example, to forbid reflections of a tile. We will do this in one example below.

The significance of boundary words is the relationship between the boundary word of a region and the boundary words of the tiles that occur in a tiling. This is given by the following (note that our statement is slightly stronger than that given by Conway and Lagarias).

THEOREM 3.3 (Conway-Lagarias). *Suppose that the simply connected region R is tiled by $T_1, T_2, \ldots, T_n$, one copy of each. Then a boundary word of R can be written as*

$$w_R = \widetilde{w}_1 \widetilde{w}_2 \cdots \widetilde{w}_n$$

where $\widetilde{w}_i$ is conjugate to a boundary word of T_i, this being an identity in the free group on the generators x and y.

Proof. We argue by induction on n. The case $n = 1$ is trivial. So suppose that $n > 1$, and that the theorem holds for all simply connected regions tiled by fewer than n tiles. Fix a tiling of R by $T_1, T_2, \ldots, T_n$, and consider one of the tiles, T, that meets the boundary of R. Suppose it meets the boundary along $k \geq 1$ segments, some of which may be isolated points. Removing T from the region results in a new region with k components, $R_1, R_2, \ldots, R_k$, some of which may touch at a corner. We label the boundary word of each R_i as $v_i^{-1} u_i$, where u_i is the word along the part of the boundary shared with R, and v_i is along the part shared with the boundary of the tile T. Let $t_1, t_2, \ldots, t_k$ be the words along the segments where T meets the boundary of R. Then we may take for a boundary word of R the element $w_R = t_1 u_1 t_2 u_2 \cdots t_k u_k$. A boundary word for T is then $w_T = t_1 v_1 t_2 v_2 \cdots t_k v_k$. (In Figure 3.4, t_2 is the empty word.)

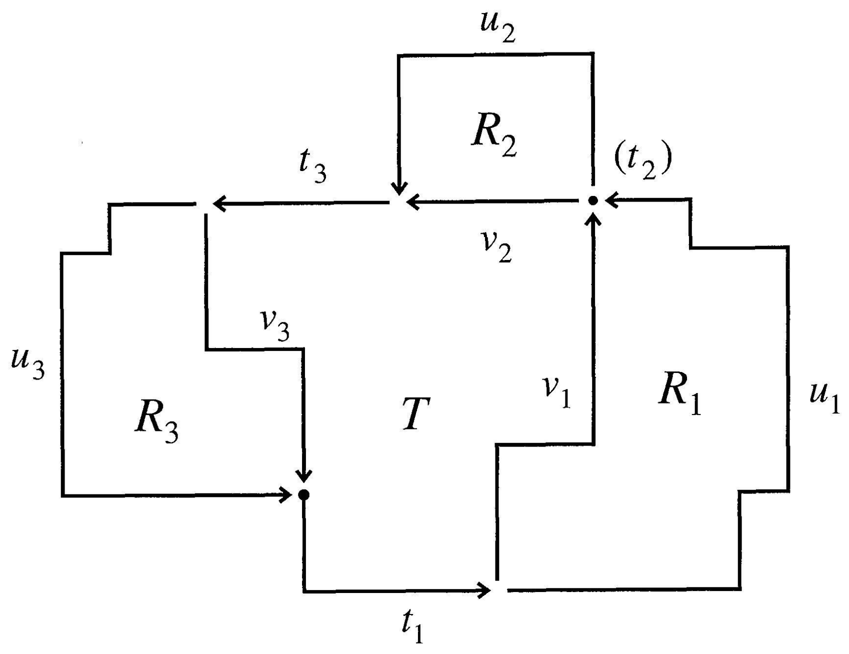


FIGURE 3.4
Decomposition of tiling

Thus we have

$$w_R = w_T \widetilde{w}_{R_1} \widetilde{w}_{R_2} \cdots \widetilde{w}_{R_k} \tag{3.5}$$

where each $\widetilde{w}_{R_i} = (t_{i+1}v_{i+1}t_{i+2}v_{i+2}\cdots t_k v_k)^{-1}(v_i^{-1}u_i)(t_{i+1}v_{i+1}t_{i+2}v_{i+2}\cdots t_k v_k)$ is a conjugate of the boundary word of R_i. The induction hypothesis applies to each R_i, and each tile occurs precisely once in T and the tilings of the R_i's. Thus (3.5) implies that

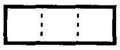
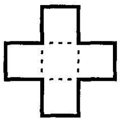
$$w_R = w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n)}$$

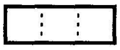
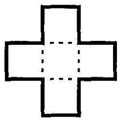
where σ is a permutation of $\{1, 2, \ldots, n\}$, and w_i is conjugate to a boundary word of T_i. It is easy to show that this implies that $w_R = \widetilde{w}_1 \widetilde{w}_2 \cdots \widetilde{w}_n$ where each $\widetilde{w}_i$ is conjugate to w_i. This completes the induction and the proof. □

An immediate consequence is the following.

COROLLARY 3.6. *Suppose that x and y are elements of a group G, such that the boundary word of every tile in $\mathcal{T}$ is the identity element of G. If a (simply connected) region can be tiled by $\mathcal{T}$, then its boundary word also gives the identity element of G.* □

REMARK 3.7. The converse of Corollary 3.6 is false in general, even if G is taken to be the "largest" group in which the boundary words of all tiles in $\mathcal{T}$ are trivial. This is due to the non-abelian analogue of signed tilings (see Corollary 6.6 below).

EXAMPLE 3.8. $\mathcal{T} = \{$, $\}$, allowing all orientations. Torsten Sillke asked if these two polyominoes could tile any rectangle whose area is not a multiple of 3. The next result shows that the answer to his query is "no".

THEOREM 3.9. *If* $\mathcal{T} = \{$, $\}$ *tiles a rectangle, then one side is divisible by* 3.

Proof. First note that it suffices to prove that $\mathcal{T}$ cannot tile any rectangle both of whose dimensions are congruent to 1 modulo 3. For if $\mathcal{T}$ tiles a $(3m+2) \times (3n+1)$ rectangle, then two of these tilings may be juxtaposed to give a tiling of a $(6m+4) \times (3n+1)$ rectangle. Similarly, if $\mathcal{T}$ tiles a $(3m+2) \times (3n+2)$ rectangle, then it also tiles a $(6m+4) \times (6n+4)$ rectangle. Thus we need only show that $\mathcal{T}$ cannot tile any $(3m+1) \times (3n+1)$ rectangle. Let x be the 3-cycle $(1,2,3) \in S_5$, and let y be the 3-cycle $(3,4,5)$. Then we easily check that $x^3yx^{-3}y^{-1} = xy^3x^{-1}y^{-3} = xyxyx^{-1}yx^{-1}y^{-1}x^{-1}y^{-1}xy^{-1} = 1$, so the boundary words of all tiles are trivial. However, the boundary word of a $(3m+1) \times (3n+1)$ rectangle is $x^{3m+1}y^{3n+1}x^{-(3m+1)}y^{-(3n+1)} = (2,3,5)$, so it cannot be tiled. $\square$

REMARK 3.10. A 1×1 square has a signed tiling by $\mathcal{T}$, so the tile homology technique cannot prove this result.

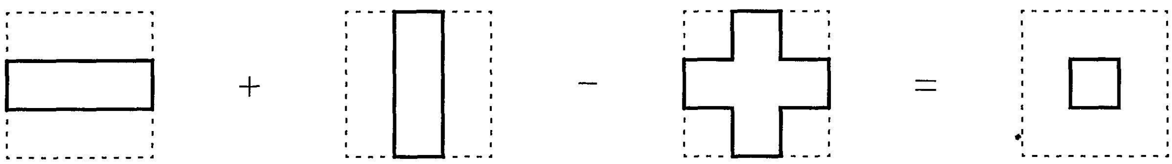


FIGURE 3.11
Signed tiling of a 1×1 square

REMARK 3.12. One might suspect that every rectangular tiling by $\mathcal{T}$ uses only the straight tromino. If this were the case, then the theorem would be somewhat less interesting, and a proof could be given by a checkerboard type argument. However, a 10×15 rectangle has a tiling by $\mathcal{T}$ which actually uses the X pentomino.

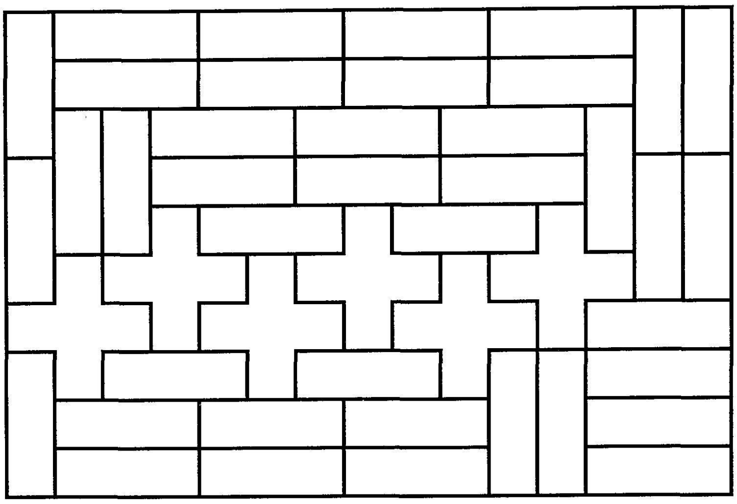


FIGURE 3.13
10×15 rectangle

QUESTION 3.14. *Is there a rectangular tiling by $\mathcal{T}$ that uses exactly three X pentominoes?*

Theorem 3.9 shows that the number of X's in a rectangular tiling must be a multiple of 3. The tiling of Figure 3.13 has 6 X's and the following tiling has 9.

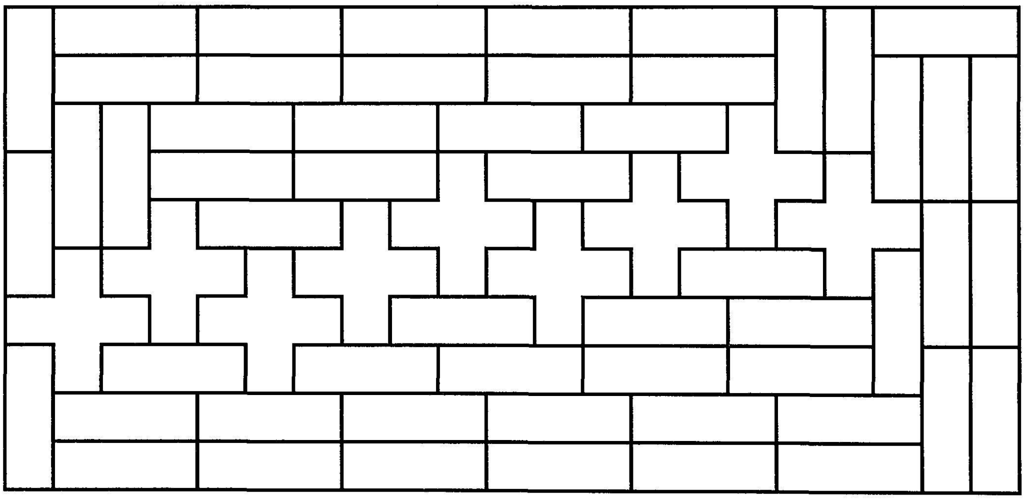


FIGURE 3.15
10×21 rectangle with nine X pentominoes

From the tilings in Figures 3.13 and 3.15, it is easy to construct rectangular tilings with $3n$ X's, for any $n \geq 2$.