L'Enseignement Mathématique, t. 50 (2004), p. 355–365

ON INTEGRAL SOLUTIONS OF QUADRATIC INEQUALITIES

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Dedicated to Professor Henri Cartan for his 100^{th} birthday

ABSTRACT. This paper studies the small values of indefinite quadratic forms with real coefficients in n variables. It shows that for $n \geq 3$ all the Markoff-type spectra of these forms consist of isolated points (apart possibly from the point 0). This improves a previous result which was obtained with much more complicated methods.

1. INTRODUCTION

The aim of the paper is to prove the following isolation theorem.

THEOREM 1.1. *Let $n \geq 3$ be an integer. Then for any $\varepsilon > 0$ and any given non-singular indefinite quadratic form f in n variables, with real coefficients, there are integers $x_1, \dots, x_n$ such that*

$$0 < f(x_1, \dots, x_n) < \varepsilon |D(f)|^{1/n} \tag{1}$$

unless f is equivalent to a positive multiple of one of a finite number of forms.

In the statement above, $D(f)$ is the determinant of f, that is, $D(f) = \det(f_{ij})$, where $f = \sum f_{ij} x_i x_j$ with $f_{ij} = f_{ji}$. As usual, two forms f and g are said to be equivalent if $f = g \circ L$ for some linear transformation L given by a unimodular matrix with integral coefficients.

By the celebrated result of Margulis (previously the Oppenheim conjecture) [4], for every indefinite irrational quadratic form f in $n \geq 3$ variables the set $f(\mathbf{Z}^n)$ is dense in $\mathbf{R}$. It follows that in order to prove Theorem 1.1 it suffices to consider only forms with rational coefficients.

A result slightly weaker than Theorem 1.1 was obtained by Vulakh [11] (he dealt with $|f|$ rather than f). His approach is entirely different from ours, more complex and using a variety of arithmetical tools, while we use the geometry of numbers.

Let $\mathcal{E}_n$ be the set of non-singular indefinite quadratic forms in n variables with coefficients in $\mathbf{R}$. Given $f \in \mathcal{E}_n$ let

$$P(f) = \inf\{\text{strictly positive values of } f(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbf{Z}^n\}$$

and

$$\alpha(f) = P(f)/|D(f)|^{1/n}.$$

Theorem 1.1 can be stated in the following equivalent form, more in the spirit of the classical Markoff Chain Theorem (see for example [1]).

THEOREM 1.2. *Given an integer $n \geq 3$, there is a decreasing sequence $(a_i)_{i\in\mathbf{N}}$ of positive rational numbers, and a sequence $(f_i)_{i\in\mathbf{N}}$ of quadratic forms in $\mathcal{E}_n$ with coefficients in $\mathbf{Z}$, such that*

(i) $a_i \to 0$ *as* $i \to \infty$;

(ii) $\alpha(f_i) = \sqrt[n]{a_i}$ *for* $i = 1, 2, \ldots$;

(iii) *each* $f \in \mathcal{E}_n$ *with* $\alpha(f) > \sqrt[n]{a_{k+1}}$ *is equivalent to a positive multiple of one of the forms* $f_1, \ldots, f_k$.

EXAMPLE 1.3. For $n = 3$, the first 6 terms of the sequence $(a_i)_{i\in\mathbf{N}}$ are known from the classical works of Davenport [5] and Watson [14]. They are $a_1 = \frac{27}{4}$, $a_2 = \frac{343}{64}$, $a_3 = \frac{125}{27}$, $a_4 = a_5 = a_6 = 4$. The corresponding forms f_i, $1 \leq i \leq 6$, are also known explicitly. For example, $f_1 = 4xy - z^2$.

EXAMPLE 1.4. For $n = 4$ the first 12 terms of the sequence $(a_i)_{i\in\mathbf{N}}$ (and the corresponding forms) are known from the work of Oppenheim [8] and Jackson [7]. They are $a_1 = 16$, $a_2 = \frac{256}{27}$, $a_3 = a_4 = \frac{16}{3}$, $a_5 = a_6 = a_7 = \frac{81}{16}$, $a_8 = a_9 = a_{10} = a_{11} = a_{12} = 4$.

EXAMPLE 1.5. For $n \geq 4$ the first term $\sqrt[n]{a_1}$ is 2 for even n and is $2^{\frac{n-1}{n}}$ for odd n (see [12]). In particular, given any f in $\mathcal{E}_n$ with $n \geq 3$, the inequality $0 < f(x) \leq 2|D(f)|^{1/n}$ can always be satisfied for some $x \in \mathbf{Z}^n$.

It follows from Theorem 1.1 that for $n \geq 3$ the non-zero points of the Markoff spectrum

$$M_n = \{\alpha(f) : f \in \mathcal{E}_n\}$$

are isolated. The spectrum M_n comes from inequalities of the type $0 < f$. More generally, there are other Markoff spectra of $\mathcal{E}_n$ associated with inequalities $0 \leq f$, $0 < |f|$ and $0 \leq |f|$. These spectra are proper subsets of M_n and thus their non-zero points are also isolated. The first 11 terms of the Markoff spectrum of $\mathcal{E}_3$ associated with the inequality $0 \leq |f|$ are given in [10]. It should be stressed that when zero is included in the inequalities of Theorem 1.1 (so that (1) becomes $0 \leq f(x_1, \ldots, x_n) < \varepsilon |D(f)|^{1/n}$) the problem becomes less difficult because the need for Lemma 2.4 below disappears.

We shall give the proof of Theorem 1.1 after some preparation in the next section. Theorem 1.2 follows directly from Theorem 1.1.

2. Preliminaries

For $f \in \mathcal{E}_n$ we shall make use of the notation $N(f)$ to denote $P(-f)$. We shall also frequently use the following inequality ([9]) linking $P(f)$ and $N(f)$.

PROPOSITION 2.1 (The Oppenheim inequality). *For $n \geq 3$*

$$(N(f))^{2n-2} \leq c_n \, (P(f))^{n-2} \, |D(f)|$$

where c_n is a constant depending only on n. For $n = 3$ we can take $c_3 = 22$.

LEMMA 2.2. *For $n \geq 3$ let $(f_k)_{k \in \mathbf{N}}$ be a sequence of quadratic forms in $\mathcal{E}_n$ such that $\lim_{k \to \infty} \alpha(f_k) > 0$. Then there is a subsequence $(f_{k_i})_{i \in \mathbf{N}}$ such that $\lim_{i \to \infty} \alpha(-f_{k_i}) > 0$.*

Proof. The Oppenheim inequality above implies that

$$(\alpha(\pm f))^{2n-2} \leq c_n \, (\alpha(\mp f))^{n-2}$$

which in turn implies the result. $\square$

We shall let $s(f)$ be the signature of the form $f \in \mathcal{E}_n$ and then we have

COROLLARY 2.3. *If Theorem 1.1 holds for all forms in $\mathcal{E}_n$ with signature s, then it also holds for all forms in $\mathcal{E}_n$ with signature $-s$.*

Proof. The corollary follows immediately from Lemma 2.2, in view of the fact that $-s(f) = s(-f)$. $\square$

Now fix a form $\varphi \in \mathcal{E}_n$. We say that a lattice Λ in $\mathbf{R}^n$ *defines the form* $f \in \mathcal{E}_n$ (with respect to φ) if, for some basis $\mathbf{a}_1, \dots, \mathbf{a}_n$ of Λ, we have

$$f(\mathbf{x}) = \varphi\left(\sum_{i=1}^{n} x_i\,\mathbf{a}_i\right), \quad \text{for each } \mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n .$$

If A is an automorph of φ and Λ defines f, then $A(\Lambda)$ also defines f. For a lattice $\Lambda \subset \mathbf{R}^n$ define

$$|\Lambda| = \min\{\|\mathbf{a}\| : \mathbf{a} \in \Lambda \setminus \{0\}\} ,$$

where $\|\mathbf{a}\| = \max\{|a_i| : 1 \le i \le n\}$ if $\mathbf{a} = (a_1, \dots, a_n)$. Also let

$$P_\varphi(\Lambda) = \inf\{\varphi(\mathbf{a}) : \mathbf{a} \in \Lambda, \varphi(\mathbf{a}) > 0\} .$$

If Λ defines f, then clearly

$$P_\varphi(\Lambda) = P(f) \quad \text{and} \quad (d(\Lambda))^2 = |D(f)| ,$$

where $d(\Lambda)$ is the determinant of Λ.

LEMMA 2.4. *Let f be a form in $\mathcal{E}_3$ with signature 1. If $P(f) = 1$ there is a lattice Λ in $\mathbf{R}^3$, defining f with respect to $\varphi = x^2 + yz$, and such that*

$$|\Lambda| \ge \frac{\beta}{2} ,$$

where $\beta = \min\left(1, (22\,|D(f)|)^{-1}\right)$.

Proof. For $n = 3$ the Oppenheim inequality mentioned above and applied to $-f$ gives

$$(P(f))^4 \le 22\,N(f)\,|D(-f)| .$$

Since $P(f) = 1$ by assumption, we obtain

$$\beta \le (22\,|D(f)|)^{-1} \le N(f) .$$

Let $Z = \varphi^{-1}(0)$ and $H = \varphi^{-1}(]-\beta, 1[) \setminus \{0\}$. The previous inequality and the equality $P(f) = 1$ imply that for *every* lattice X in $\mathbf{R}^3$ defining f we have

$$X \cap (H \setminus Z) = \varnothing .$$

If $\mathbf{w} \in \mathbf{R}^3 \setminus H$ and $\mathbf{w} \ne \mathbf{0}$, then clearly $\|\mathbf{w}\| \ge \sqrt{\beta/2} > \beta/2$. Hence, in order to prove the lemma, we only have to check that for *some* lattice Λ defining f we have $\|\mathbf{w}\| \ge \beta/2$ for every non-zero $\mathbf{w} \in \Lambda \cap Z$.

Let X be any lattice in $\mathbf{R}^3$ defining f. Since $P(f) = 1$, and therefore f is necessarily a rational form by the Margulis theorem, it follows that f takes the value 1 at some point in $\mathbf{Z}^3$. This, together with the fact that $\varphi(1,0,0) = 1$, implies the existence of an automorph B of φ such that the point $\mathbf{v} = (1,0,0)$ is in $B(X)$ (see [13], p. 11, Theorem 5). Therefore, replacing X by $B(X)$, we can assume from the beginning that $(1,0,0) \in X$.

Let $\mathbf{w} \in X \cap Z$. Assume first that $\mathbf{w}$ is not orthogonal to $\mathbf{v} = (1,0,0)$. Then $\mathbf{w} = (a,b,c)$ with $a \neq 0$. We claim that $\|\mathbf{w}\| \geq \frac{1}{2} \geq \beta/2$. Replacing $\mathbf{w}$ by $-\mathbf{w}$, if necessary, we may assume that $a < 0$. If $|a| < \frac{1}{2}$ then

$$0 < \varphi(\mathbf{w}+\mathbf{v}) = (1+a)^2 + bc = 1 + 2a < 1\,;$$

in other words $w + v \in X \cap (H \setminus Z)$, contradicting the fact that $X \cap (H \setminus Z)$ is empty.

So if X is any lattice defining f and containing $\mathbf{v} = (1,0,0)$ then for each non-zero $\mathbf{w}$ in X we have $\|\mathbf{w}\| \geq \beta/2$, except possibly when $\mathbf{w} \in X \cap Z$ and $\mathbf{w}$ is orthogonal to $\mathbf{v}$.

We shall now deal with the case where $\mathbf{w} \in X \cap Z$, $\mathbf{w} \neq \mathbf{0}$, and $\mathbf{w}$ is orthogonal to $\mathbf{v}$. Then $\mathbf{w}$ must be of the form $(0,t,0)$ or $(0,0,t)$ for some $t \neq 0$. (If no such vector exists the proof is finished.)

If there are two points $\mathbf{w}_1$ and $\mathbf{w}_2$ in $X \cap Z$ that are orthogonal to $\mathbf{v}$ and linearly independent, we can choose these points to be $\mathbf{w}_1 = (0,t,0)$ and $\mathbf{w}_2 = (0,0,t')$ with, say, $0 < t \leq t'$ and t and t' least possible. Then necessarily $\delta = tt' \geq 1$, as otherwise $0 < \varphi(\mathbf{w}_1 + \mathbf{w}_2) = tt' < 1$ and $\mathbf{w}_1 + \mathbf{w}_2$ would be in $X \cap (H \setminus Z)$ which is empty. Let A be an automorph of φ defined by

$$A(x,y,z) = (x, y/t, tz)\,.$$

Then $\Lambda = A(X)$ is a lattice defining f with $\mathbf{v} \in \Lambda$ and, for each $\mathbf{w} \in \Lambda \cap Z$, we have either $\|\mathbf{w}\| \geq 1$ if $\mathbf{w}$ is of the form $(0,u,0)$ or $\|\mathbf{w}\| \geq \delta \geq 1$ if $\mathbf{w}$ is of the form $(0,0,u)$. Hence, in the case under consideration, for each non-zero $\mathbf{w} \in \Lambda \cap Z$ that is orthogonal to $\mathbf{v}$, we have $\|\mathbf{w}\| \geq 1 > \beta/2$. In other words, Λ is a lattice with the required properties: it defines f and $|\Lambda| \geq \beta/2$.

Finally, if each vector in $X \cap Z$, orthogonal to $\mathbf{v}$, is a multiple of a single vector $\mathbf{w} = (0,t,0)$ (or $\mathbf{w} = (0,0,t)$), for some $t \neq 0$, then taking the automorph A of φ defined by $A(x,y,z) = (x, y/t, tz)$ (or $(x, ty, z/t)$), we again obtain a lattice $\Lambda = A(X)$ with the required properties. $\square$

LEMMA 2.5. *Let Λ_i be a sequence of lattices in $\mathbf{R}^n$ converging to a lattice Λ and let φ be in $\mathcal{E}_n$. Then*

$$\limsup_{i \to \infty} P_\varphi(\Lambda_i) \leq P_\varphi(\Lambda)\,.$$

Proof. Let $M = \varphi^{-1}((-\infty, 0])$ and take $\mathbf{a} \in \Lambda \setminus M$. Then there are points $\mathbf{a}_i \in \Lambda_i$ such that $\mathbf{a}_i \to \mathbf{a}$ as $i \to \infty$. Since $\mathbf{a}_i$ is not in M for all large enough i, we have

$$\varphi(\mathbf{a}_i) \geq \inf\{\varphi(\mathbf{y}) : \mathbf{y} \in \Lambda_i \setminus M\} = P_\varphi(\Lambda_i)$$

for all large i. By the continuity of φ we therefore have

$$\varphi(\mathbf{a}) \geq \limsup_{i \to \infty} P_\varphi(\Lambda_i)\,.$$

Since $\mathbf{a}$ was an arbitrary point of $\Lambda \setminus M$, the lemma follows. $\square$

3. Proof of Theorem 1.1

We shall prove Theorem 1.1 in two steps. First for $n = 3$, then, using induction, for $n \geq 4$.

3.1 Case $n = 3$

If Theorem 1.1 is false for $n = 3$, then there is an infinite sequence f_m of ternary forms in $\mathcal{E}_3$, having the same signature, such that no f_m is equivalent to a multiple of f_q for $q \neq m$, and such that

$$\alpha(f_m) \to a > 0 \text{ as } m \to \infty\,.$$

Moreover, replacing f_m by $-f_m$ if necessary, and taking possibly a subsequence, we can assume without loss of generality (using Lemma 2.2) that all the f_m have signature 1. Then we scale each f_m to have

$$P(f_m) = 1$$

and thus

$$|D(f_m)| \to a^{-3} \text{ as } m \to \infty\,. \tag{2}$$

Let Λ_m be a lattice in $\mathbf{R}^3$ defining f_m with respect to $\varphi = x^2 + yz$. In particular, $d(\Lambda_m) = |D(f_m)|^{\frac{1}{2}}$ and $P_\varphi(\Lambda_m) = 1$. It follows from (2) that for each m

$$d(\Lambda_m) \leq \gamma \text{ for some } \gamma > 0\,. \tag{3}$$

From (3) and Lemma 2.4, we can assume that for some η (for example $\eta = \min(\frac{1}{2}, \frac{1}{44\gamma^2})$) we have

$$0 < \eta \leq |\Lambda_m| \tag{4}$$

for every m.

By Mahler's compactness theorem ([2], p. 137, Theorem IV), properties (3) and (4) imply that $\{\Lambda_m\}_{m \in \mathbf{N}}$ contains a subsequence $\Gamma_i = \Lambda_{m_i}$ converging to a lattice Γ. Let B be a basis of Γ and let B_i be a basis of Γ_i such that B_i converges to B. Let g_i be a quadratic form in $\mathcal{E}_3$ which is defined, with respect to φ, by (Γ_i, B_i). Similarly let g be a quadratic form in $\mathcal{E}_3$ which is defined, with respect to φ, by (Γ, B). Then the sequence of forms g_i converges to the form g. Since g_i is equivalent to f_{m_i}, we have $P_\varphi(\Gamma_i) = P(g_i) = 1$. By Lemma 2.5

$$P(g) = P_\varphi(\Gamma) \geq \limsup P_\varphi(\Gamma_i) = 1$$

which implies that g is a multiple of a rational form (by Margulis' theorem again).

By the Cassels-Swinnerton-Dyer theorem ([3], p. 86, Theorem 8), for each g_i close enough to g, there is an $\mathbf{x}_i \in \mathbf{Z}^3$ with $0 < g_i(\mathbf{x}_i) < 1$, contradicting the fact that $P(g_i) = 1$.

This completes the proof of the case $n = 3$.

3.2 CASE $n \geq 4$

First we need a preliminary result about binary quadratic forms.

LEMMA 3.1. *Let F be a form in $\mathcal{E}_2$. Then for each $n \geq 3$*

$$P(F)^n N(F)^{n-2} \leq |4D(F)|^{n-1}. \tag{5}$$

Proof. If F does not represent zero on $\mathbf{Z}^2$, except trivially, the inequality is implied by stronger results in [6], Theorems 2–4. When F represents zero non-trivially we may scale it to have determinant -1 and then, by an integral unimodular transformation, suppose that it has the shape

$$2xy - \theta y^2$$

where $0 \leq \theta < 1$. This gives $P(F) \leq 2 - \theta$ and either $N(F) = 2$ if $\theta = 0$ or $N(F) \leq \theta$. Then either $P(F)^n N(F)^{n-2} = 2^{2n-2}$ if F is equivalent to $2xy$ or $P(F)^n N(F)^{n-2} \leq (2-\theta)^n \theta^{n-2} < 2^n$ otherwise. Thus (5) holds in all cases. □

We shall now prove Theorem 1.1 for $n \geq 4$. We suppose that the theorem has already been established for indefinite forms in $n-1$ variables and we shall prove it for n variables using induction on n. It suffices, without losing generality, to consider the case of forms in $\mathcal{E}_n$ having signature less than $n-2$. The remaining case of signature $n-2$ would follow from Corollary 2.3.

For a given $\varepsilon > 0$ we shall concentrate on establishing

$$P(f)^n < \varepsilon|D(f)| \tag{6}$$

where $f \in \mathcal{E}_n$ and $s(f) < n-2$. This is a convenient alternative to the original inequality in Theorem 1.1. By the Margulis Theorem it suffices to consider only rational forms. For such a form f we have $P(f) > 0$ and $N(f) > 0$ and we then scale f to have

$$N(f) = 1\,.$$

Since f is rational it takes the value -1 and an integral unimodular transformation puts it in the shape

$$-(x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n)^2 + g(x_2, \ldots, x_n)\,.$$

Here g is a rational form in $n-1$ variables which is indefinite, the signature of f being by assumption less than $n-2$. Also

$$|D(f)| = |D(g)|\,.$$

The form g attains its least positive value $P(g) = g(l_2, \ldots, l_n) = v$, say, for some $l_i \in \mathbf{Z}$. The binary section F of f defined by

$$F(x, y) = f(x, l_2 y, l_3 y, \ldots, l_n y)$$

is then an indefinite non-singular binary form and so satisfies the inequality in (5). Since f represents all the values of F we have

$$P(f) \leq P(F) \quad \text{and} \quad N(f) = N(F) = 1$$

and hence

$$P(f)^n \leq P(F)^n N(F)^{n-2} \leq |4D(F)|^{n-1} = (4v)^{n-1}\,. \tag{7}$$

The case $n-1$ of the theorem implies that we will have

$$v^{n-1} < \varepsilon 4^{1-n}|D(g)|$$

unless g is equivalent to a positive multiple of one of a finite number of forms. Apart from those possibilities, (7) gives the desired inequality

$$P(f)^n < \varepsilon|D(g)| = \varepsilon|D(f)|\,.$$

So either (6) holds or for some $k > 0$ the form f is equivalent to

$$f'(x_1, \ldots, x_n) = -(x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n)^2 + kg'(x_2, \ldots, x_n) \tag{8}$$

where

$$g'(x_2,\dots,x_n) = (x_2+\theta_3x_3+\cdots)^2 + a_3(x_3+\cdots)^2+\cdots+a_nx_n^2\,,$$

$|D(f)| = |D(f')| = k^{n-1}|D(g')|$ and there are only a finite number of choices for the form g' with $g'(1,0,\dots,0)=1$.

Observe that by simple parallel transformations on x_1, and thus without changing kg', we can assume that $\alpha_2,\dots,\alpha_n$ satisfy $0\le\alpha_i<1$. The strategy for ending the proof is to show that for each g' in (8) there are only a finite number of allowable values for $k,\alpha_2,\dots,\alpha_n$ with $0\le\alpha_i<1$.

The form f' represents $k-(x_1+\alpha_2)^2$ and for a suitable x_1 we can have

$$k-1\le k-(x_1+\alpha_2)^2\le k-\frac{1}{4}\,.$$

This contradicts $N(f')=N(f)=1$ unless $4k\ge 1$.

For each choice of g' the leading $(n-1)$–ary section $f'(x_1,x_2,\dots,x_{n-1},0)$ of f' has determinant $-k^{n-2}D(g')/a_n\ne 0$ and so, for any $\varepsilon_1>0$, it will represent a positive value $v_1<\varepsilon_1\left|k^{n-2}D(g')/a_n\right|^{1/n-1}$ unless it is equivalent to a multiple of one of a finite number of forms. Using $4k\ge 1$ and choosing $\varepsilon_1\le\frac{1}{2}\varepsilon|D(g')/a_n|^{-1/n-1}$ the inequality for v_1 implies

$$v_1<2\varepsilon_1|D(g')/a_n|^{1/n-1}k\le\varepsilon k$$

so taking

$$\varepsilon_1=\min\left(\frac{1}{2}\varepsilon|D(g')/a_n|^{-1/n-1},\,a_n^{1/n-1}\right)$$

we would have

$$P(f)^n\le v_1^n<\left|\varepsilon k^{n-1}D(g')\right|=\varepsilon|D(f')|=\varepsilon|D(f)|\,.$$

Hence, if $P(f)^n<\varepsilon|D(f)|$ fails, $f'(x_1,\dots,x_{n-1},0)$ must be equivalent to a positive multiple of one of a finite list of $(n-1)$-ary forms $h_1,\dots,h_t$ which we can take to be normalised in any way we please. Moreover, for each form h_i there can only be one multiple r_ih_i which makes $f'(x_1,x_2,\dots,x_{n-1},0)$ equivalent to r_ih_i because there will only be one value of r_i making

$$N(r_ih_i)=N(f'(x_1,x_2,\dots,x_{n-1},0))=1\,.$$

This means that for each h_i there will only be one allowable value of k giving $D(f'(x_1,x_2,\dots,x_{n-1},0)) = -k^{n-2}D(g')/a_n = r_i^{n-1}D(h_i)$. The number of allowable values of k in (8) is therefore finite.

So, if $P(f)^n<\varepsilon|D(f)|$ fails, there are only a finite number of possibilities for the form kg' in (8) and for each of these possibilities $f'(x_1,x_2,\dots,x_{n-1},0)$

must be equivalent to one of a finite number of forms $r_i h_i$. Now let q be the least common denominator of the coefficients of $r_i h_i$. If we could have $f'(x_1, x_2, \ldots, x_{n-1}, 0)$ equivalent to $r_i h_i$ for an infinite number of possible values of α_2, with $0 \leq \alpha_2 < 1$, there would be two allowable values, say β and γ, with $0 < |\beta - \gamma| < \frac{1}{2q}$. Then considering $f'(0,1,0,\ldots,0)$, we see that $f'(x_1, x_2, \ldots, x_{n-1}, 0)$ represents $k - \beta^2$ for $\alpha_2 = \beta$ and represents $k - \gamma^2$ for $\alpha_2 = \gamma$. However

$$\left|(k - \beta^2) - (k - \gamma^2)\right| = \left|\beta^2 - \gamma^2\right| < \frac{1}{2q}|\beta + \gamma| < \frac{1}{q}$$

contradicting the fact that distinct values of $r_i h_i$ are never closer than $\frac{1}{q}$. Similar considerations of $f'(0,0,1,\ldots,0), \ldots, f'(0,0,\ldots,1,0)$ show that there are only a finite number of allowable values of $\alpha_3, \ldots, \alpha_{n-1}$ for each $r_i h_i$.

Finally, to show that the number of allowable values of α_n in (8) is finite, consider the indefinite $(n-1)$-ary sections $f'(x_1, x_2, \ldots, x_{n-2}, 0, x_n)$, $f'(x_1, x_2, \ldots, x_{n-2}, x_n, x_n)$ and $f'(x_1, x_2, \ldots, x_{n-2}, 2x_n, x_n)$. At least one of these, called $\psi(x_1, x_2, \ldots, x_{n-2}, x_n)$ say, has a non-zero determinant (whose value depends only on k and the coefficients of g'). So, taking $\varepsilon_2 = \varepsilon\, |D(f)|^{1/n}\, |D(\psi)|^{-1/n-1} > 0$, ψ will represent a small positive value

$$v_2 < \varepsilon_2\, |D(\psi)|^{1/n-1} = \varepsilon\, |D(f)|^{1/n}$$

unless it is equivalent to a multiple of one of a finite number of forms. Since $N(\psi) = 1$, there will again only be one allowable multiple for each of the finite number of forms. As before, we can also see that for each of the finite number of possibilities for $\alpha_2, \ldots, \alpha_{n-1}, k, g'$ there will only be a finite number of allowable values of α_n.

It follows that the number of forms $f \in \mathcal{E}_n$ for which (6) fails (for a given $\varepsilon > 0$) is finite. So the theorem holds for n-ary forms.

REFERENCES

[1] CASSELS, J. W. S. *An Introduction to Diophantine Approximation.* Camb. Univ. Press, Cambridge, 1957.

[2] —— *An Introduction to the Geometry of Numbers.* Springer, Berlin, 1971.

[3] CASSELS, J. W. S. and H. P. F. SWINNERTON-DYER. On the product of three homogeneous linear forms and indefinite ternary quadratic forms. *Phil. Trans. Roy. Soc. London 248* (1955), 73–96.

[4] DANI, S. G. and G. MARGULIS. Values of quadratic forms at integral points: an elementary approach. *L'Enseign. Math. (2) 36* (1990), 143–174.

[5] DAVENPORT, H. On indefinite ternary quadratic forms. *Proc. London Math. Soc. (2) 51* (1950), 145–160.

[6] JACKSON, T. H. Small positive values of indefinite binary quadratic forms. *J. London Math. Soc. 43* (1968), 730–738.

[7] —— Small positive values of indefinite quadratic forms. *J. London Math. Soc. (2) 1* (1969), 643–659.

[8] OPPENHEIM, A. One-sided inequalities for quadratic forms: II quaternary forms. *Proc. London Math. Soc. (3) 3* (1953), 418–429.

[9] —— Values of quadratic forms (I), (II). *Quart. J. Math. Oxford Ser. (2) 4* (1953), 54–66.

[10] VENKOV, B. A. On the extremal problem of Markov for indefinite ternary quadratic forms. *Izv. Akad. Nauk SSSR 9* (1945), 429–494.

[11] VULAKH, L. Y. On minima of indefinite rational quadratic forms. *J. Number Theory 21* (1985), 275–285.

[12] WATSON, G. L. One-sided inequalities for integral quadratic forms. *Quart. J. Math. Oxford Ser. (2) 9* (1958), 99–108.

[13] —— *Integral Quadratic Forms*. Camb. Univ. Press, Cambridge, 1960.

[14] —— Asymmetric inequalities for indefinite quadratic forms. *Proc. London Math. Soc. (3) 18* (1968), 95–113.

(Reçu le 20 avril 2004)

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