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ELEMENTARY CONSTRUCTION OF EXHAUSTING SUBSOLUTIONS OF ELLIPTIC OPERATORS

by Terrence NAPIER *) and Mohan RAMACHANDRAN †)

ABSTRACT. By a theorem of Greene and Wu [GreW], a noncompact connected Riemannian manifold admits a C^∞ strictly subharmonic exhaustion function. Demailly provided an elementary proof of this fact in [D]. A further simplification of Demailly's proof and some (mostly known) applications are described. Applications include the fact that the holomorphic line bundle associated to a nontrivial effective divisor on a compact connected complex manifold X admits a C^∞ Hermitian metric with positive scalar curvature.

0. INTRODUCTION

Let (M, g) be a Riemannian manifold of dimension n. The *Laplace operator* Δ_g for g is given in local coordinates $(x_1, \dots, x_n)$ by

$$\Delta_g \varphi = \frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[g^{ij} \sqrt{G} \frac{\partial \varphi}{\partial x_j} \right],$$

for every function φ of class C^2; where

$$G = \det(g_{ij}) \qquad \text{and} \qquad (g^{ij}) = (g_{ij})^{-1}.$$

A C^2 real-valued function φ is called *subharmonic* (*strictly subharmonic*) with respect to g if $\Delta_g \varphi \geq 0$ (respectively, $\Delta_g \varphi > 0$). A real-valued function ρ on a topological space X is called an *exhaustion function* if

$$\{ x \in X \mid \rho(x) < a \} \subset\subset X \qquad \forall a \in \mathbf{R}.$$

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A special case of the main result of this paper is the following:

THEOREM 0.1 (Greene and Wu [GreW]). *A connected noncompact Riemannian manifold M admits a C^∞ strictly subharmonic exhaustion function.*

Greene and Wu actually produced a proper embedding by harmonic functions and obtained the above as a consequence. Thus their proof is not elementary. A related construction is that of Ohsawa [O] of a strongly n-convex exhaustion function on an n-dimensional complex space with no compact irreducible components. Demailly [D] provided an elementary (and relatively simple) proof of Theorem 0.1 (his proof is written for the case of the Laplace operator of a Hermitian metric, but it can be modified to give the above theorem). His method is a version of the classical idea in Runge theory in one complex variable of pushing singularities to infinity using a local construction. His local construction, although short, requires some calculations which are not completely transparent.

Theorem 0.1 is of some importance. It implies, for example, that the top homology of an open manifold vanishes (cf. Theorem 2.2 below). Demailly gave his version for applications to q-convex spaces. It can also be used to give a simple proof of the Behnke-Stein approximation theorem for open Riemann surfaces. We hope to return to this question elsewhere.

In this paper, we give a local construction which is very simple and transparent. It is based on the following observation which is of some independent interest.

BUMP FUNCTION LEMMA. *Let B be a domain in M, let K be a compact subset of B, and let W be a nonempty open subset of $B\setminus K$. Then there exists a nonnegative C^∞ function α on M such that $\alpha\equiv 0$ on $M\setminus B$, $\alpha>0$ and $\Delta_g\alpha>0$ on K, and $\Delta_g\alpha\geq 0$ on $M\setminus W$.*

To conclude this introduction, we give an outline of the ideas in the proof of the main theorem. For the bump function lemma, we may assume without loss of generality that $W\subset\subset B\subset\subset M$ and we may fix a domain U and a nonnegative C^∞ function ρ on M such that

$$K\cup\overline{W}\subset U\subset\subset B,\quad \rho>0 \text{ on } \overline{U},\ \text{ and }\quad \rho<0 \text{ on } M\setminus B.$$

Replacing ρ by an approximating Morse function (see, for example, [GoG]), we may also assume that ρ has only isolated critical points in B. Fix a regular value $\epsilon > 0$ for ρ with $\rho > \epsilon$ on $\bar{U}$ and let V be the component of $\{x \in M \mid \rho(x) > \epsilon\}$ containing U. Thus $U \subset\subset V \subset\subset B$.

We will say that a mapping $\Phi\colon N \to N$ of a connected smooth manifold N onto itself has *compact support* if Φ is equal to the identity outside a compact set. Given two points $p, q \in N$, there exists a C^∞ diffeomorphism $\Phi\colon N \to N$ with compact support such that $\Phi(p) = q$ (for the set of points q in N to which p can be moved by such a diffeomorphism is open and closed). For distinct points $p_1, \dots, p_m, q_1, \dots, q_m$ in N, one gets such a Φ with $\Phi(p_j) = q_j$ for $j = 1, \dots, m$ by forming a compactly supported diffeomorphism Φ_j of $N \setminus \{p_1, \dots, \hat{p}_j, \dots, p_m, q_1, \dots, \hat{q}_j, \dots, q_m\}$ moving p_j to q_j for each $j = 1, \dots, m$ and letting Φ be the composition of the extensions by the identity for $\Phi_1, \dots, \Phi_m$.

Thus we may move into W the critical points of ρ in V by a diffeomorphism with compact support in V, and hence we may assume that $\nabla\rho \neq 0$ at each point in $\bar{V} \setminus W \supset K$. For $R > 0$, let $\beta \equiv e^{R\rho} - e^{R\epsilon}$. Then

$$\Delta_g \beta = R e^{R\rho}(\Delta_g \rho + R|\nabla\rho|^2) > 0$$

on $\bar{V} \setminus W$, provided $R \gg 0$. Finally, fixing a C^∞ function $\chi\colon \mathbf{R} \to \mathbf{R}$ such that $\chi(t) = 0$ for $t \leq 0$ and $\chi'(t) > 0$ and $\chi''(t) \geq 0$ for $t > 0$, we get a nonnegative C^∞ function

$$\alpha \equiv \begin{cases} \chi(\beta) & \text{on } V, \\ 0 & \text{on } M \setminus V. \end{cases}$$

On $V \setminus W$, we have $\alpha > 0$ and

$$\Delta_g \alpha = \chi'(\beta)\Delta_g \beta + \chi''(\beta)|\nabla\beta|^2 > 0.$$

It follows that α has the required properties.

For B a coordinate n-dimensional rectangle (or another nice set), one can easily construct such a function α explicitly (see Lemmas 1.6–1.8). Moreover, such bump functions are all that are needed to reduce the construction of a strictly subharmonic exhaustion function to point set topology (so the proof is very elementary).

One pushes the bad set off to infinity (in the usual way) as follows. Given a point $p \in M$, there is a locally finite sequence of relatively compact domains $\{B_\nu\}_{\nu=1}^\infty$ with

$$p \in B_1 \text{ and } B_\nu \cap B_{\nu+1} \neq \varnothing \text{ for } \nu = 1, 2, 3, \dots.$$

Hence there exist nonempty disjoint open sets $\{W_\nu\}_{\nu=0}^\infty$ such that $p \in N_p \equiv W_0 \subset\subset B_1$ and $W_\nu \subset\subset B_\nu \cap B_{\nu+1}$ for $\nu = 1, 2, 3, \dots$ and, as in the lemma,

C^∞ bump functions $\{\alpha_\nu\}_{\nu=1}^\infty$ such that, for each $\nu = 1, 2, 3, \ldots$, we have $\operatorname{supp}\alpha_\nu \subset B_\nu$, $\alpha_\nu > 0$ and $\Delta_g \alpha_\nu > 0$ on $\overline{W}_{\nu-1}$, and $\Delta_g \alpha_\nu \geq 0$ on $M \setminus W_\nu$. For constants $0 < r_1 \ll r_2 \ll r_3 \ll \cdots$, we get a C^∞ subharmonic function

$$\beta_p \equiv \sum_{\nu=1}^{\infty} r_\nu \alpha_\nu$$

with $\operatorname{supp}\beta_p \subset Q_p \equiv \bigcup_{\nu=1}^\infty B_\nu$ and $\beta_p > 0$ and $\Delta_g \beta_p > 0$ on the neighborhood N_p of p. Paracompactness implies that we can form a locally finite covering $\{N_{p_j}\}$ of M by such sets and a corresponding locally finite collection $\{Q_{p_j}\}$. Thus, for $R_j \gg 0$ for $j = 1, 2, 3, \ldots$, we get a C^∞ strictly subharmonic exhaustion function

$$\varphi \equiv \sum_{j=1}^{\infty} R_j \beta_{p_j} .$$

In fact, the above arguments actually give the following analogue of Urysohn's lemma:

THEOREM 0.2 (cf. Theorem 1.13). *Suppose U is a domain in a connected noncompact Riemannian manifold M, C is a connected noncompact closed subset of M with $C \subset U$, and ρ is a positive continuous function on M. Then there exists a nonnegative C^∞ subharmonic function φ on M such that $\varphi \equiv 0$ on $M \setminus U$ and $\varphi > \rho$ and $\Delta_g \varphi > \rho$ on C.*

REMARK. The existence of such a set $C \subset U$ is a necessary condition (see the remarks following Theorem 1.13). In the terminology of [EM], U has an *exit to* ∞ (relative to M).

A detailed proof of Theorem 0.1 (in fact, a proof of the existence of an exhausting subsolution for a more general elliptic operator) appears in Section 1. Some (mostly known) applications are described in Section 2. These include the fact that the holomorphic line bundle associated to a nontrivial effective divisor on a compact connected complex manifold X admits a C^∞ Hermitian metric with positive scalar curvature (Theorem 2.3).

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1. Construction of exhausting subsolutions

Throughout this section, M will denote a connected smooth manifold of dimension n and A will denote a second order linear elliptic differential operator with continuous coefficients. Thus, in local coordinates $(x_1, \ldots, x_n)$,

$$A = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c\,;$$

where a_{ij} for each i and j, b_i for each i, and c are continuous real-valued functions and (a_{ij}) is a symmetric matrix-valued function with positive eigenvalues at each point.

Theorem 0.1 follows from the theorem below if we choose for A the Laplacian Δ_g of a Riemannian metric g and, for ρ, a positive continuous exhaustion function.

THEOREM 1.1. *If M is noncompact and ρ is a positive continuous function on M, then there exists a C^∞ function φ on M such that $\varphi > \rho$ and $A\varphi > \rho$.*

The main step in the proof is the following:

PROPOSITION 1.2. *Suppose K is a compact subset of M, U is a component of $M \setminus K$ which is not relatively compact in M, and $p \in U$. Then there exists a C^∞ function α such that*

(i) *$\alpha \geq 0$ and $A\alpha \geq 0$ on M,*
(ii) $\operatorname{supp} \alpha \subset U$,
(iii) *$\alpha(p) > 0$, and*
(iv) *$A\alpha(p) > 0$.*

REMARK. Theorem 1.1 also holds for a second order locally uniformly elliptic linear differential operator A with locally bounded (not necessarily continuous) coefficients. One applies the corresponding version of Proposition 1.2 in which the property (iv) is replaced with $A\alpha > 1$ on a neighborhood of p. The generalizations considered in this paper (Theorem 1.10 and Theorem 1.13) also hold for such an operator A.

The following equivalent version of Proposition 1.2 implies that a compact set which is *topologically Runge* is convex with respect to functions α satisfying $A\alpha \geq 0$:

PROPOSITION 1.3. *Let K be a compact subset of M whose complement has no relatively compact components. Then, for each point $p \in M \setminus K$, there is a C^∞ nonnegative function α on M such that $A\alpha \geq 0$ on M, $\alpha \equiv 0$ on K, $\alpha(p) > 0$, and $A\alpha(p) > 0$.*

REMARK. If the coefficients are (for example) C^1 and the constant term $c \leq 0$ (for example, if $A = \Delta_g$), then a nonconstant subsolution on a domain cannot attain a positive maximum and, therefore, the converse will also hold. That is, if such a function α exists for some point $p \in M \setminus K$, then the component of $M \setminus K$ containing p is not relatively compact.

Proposition 1.2 and Proposition 1.3 together with standard arguments in Runge theory give Theorem 1.1. Proofs are provided for the convenience of the reader. For this, we need two elementary observations (cf. Malgrange [M] or Narasimhan [N]).

LEMMA 1.4. *Let X be a noncompact, connected, locally connected, locally compact, Hausdorff topological space. If K is a compact subset of X and $\widehat{K}$ is the union of K with all of the relatively compact components of $X \setminus K$, then $\widehat{K}$ is compact, $X \setminus \widehat{K}$ has only finitely many components, and each component of $X \setminus \widehat{K}$ has noncompact closure.*

Proof. We may assume without loss of generality that $K \neq \varnothing$. Since X is Hausdorff, K is closed and, since X is locally connected, the components of $X \setminus K$ are open. It follows that $\widehat{K}$ is a closed set whose complement has no relatively compact components (since $X \setminus \widehat{K}$ is the union of components of $X \setminus K$ with noncompact closure).

Since X is locally compact Hausdorff, we may choose a relatively compact neighborhood Ω of K in X. The components of $X \setminus K$ are open and disjoint, so only finitely many meet the compact set $\partial\Omega \subset X \setminus K$. By replacing Ω by the union of Ω with all relatively compact components of $X \setminus K$ meeting $\partial\Omega$, we may assume that no relatively compact component of $X \setminus K$ meets $\partial\Omega$. On the other hand, every component E of $X \setminus K$ must satisfy

$$\bar{E} \cap K = \partial E \neq \varnothing .$$

For E is open and closed relative to $X \setminus K$, so $\partial E \subset K$, while $E \neq X$, so $\partial E = \bar{E} \setminus E \neq \varnothing$ (E cannot be both open and closed in the connected space X). It follows that, if E meets $X \setminus \Omega$, then E meets $\partial\Omega$ and hence E is *not* relatively compact in X. Thus

$$X \setminus \Omega \subset E_1 \cup \cdots \cup E_m$$

for finitely many components $E_1, \ldots, E_m$ of $X \setminus K$, none relatively compact in X, and $\widehat{K} \subset \Omega \subset\subset X$. The claim now follows. □

LEMMA 1.5. *Let X be a second countable, noncompact, connected, locally connected, locally compact, Hausdorff topological space. Then there is a sequence of compact sets $\{K_\nu\}_{\nu=1}^\infty$ such that $X = \bigcup_{\nu=1}^\infty K_\nu$ and, for each ν, $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ and $\widehat{K}_\nu = K_\nu$, where $\widehat{K}_\nu$ is defined as in Lemma 1.4.*

Proof. We may choose a sequence of compact sets $\{H_\nu\}$ such that $X = \bigcup_{\nu=1}^\infty H_\nu$. We set $K_1 = \widehat{H_1}$. Given K_ν, we may choose a compact set $K'_{\nu+1}$ such that $H_\nu \cup K_\nu \subset \overset{\circ}{K'}_{\nu+1}$ and set $K_{\nu+1} = \widehat{K'_{\nu+1}}$. This yields the desired sequence. □

Proof of Theorem 1.1. By Lemma 1.5, we may choose a sequence of nonempty compact sets $\{K_\nu\}$ such that $M = \bigcup_{\nu=1}^\infty K_\nu$ and, for each ν, $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ and $M \setminus K_\nu$ has no relatively compact components. Set $K_0 = \varnothing$.

Given $p \in M$, there is a unique $\nu = \nu(p)$ with $p \in K_{\nu+1} \setminus K_\nu$ and we may apply Proposition 1.3 to get a C^∞ nonnegative function α_p and a relatively compact neighborhood V_p of p in $M \setminus K_\nu$ such that $A\alpha_p \geq 0$ on M, $\alpha_p \equiv 0$ on K_ν, and $\alpha_p > \rho$ and $A\alpha_p > \rho$ on V_p (one obtains the last two conditions by multiplying by a sufficiently large positive constant). Thus we may choose a sequence of points $\{p_k\}$ in M and corresponding functions $\{\alpha_{p_k}\}$ and neighborhoods $\{V_{p_k}\}$ so that $\{V_{p_k}\}$ forms a locally finite covering of M (for example, one may take $\{p_k\}$ to be an enumeration of the countable set $\bigcup_{\nu=0}^\infty Z_\nu$ where, for each ν, Z_ν is a finite set of points in $M \setminus \overset{\circ}{K}_\nu$ such that $\{V_p\}_{p \in Z_\nu}$ covers $K_{\nu+1} \setminus \overset{\circ}{K}_\nu$). The collection $\{\operatorname{supp} \alpha_{p_k}\}$ is then locally finite in M since $\operatorname{supp} \alpha_{p_k} \subset M \setminus K_\nu$ whenever $p_k \notin K_\nu$. Hence the sum $\sum_{k=1}^\infty \alpha_{p_k}$ is locally finite and, therefore, convergent to a C^∞ function φ on M satisfying $\varphi \geq \alpha_{p_k} > \rho$ and $A\varphi \geq A\alpha_{p_k} > \rho$ on V_{p_k} for each k. Therefore, since $\{V_{p_k}\}$ covers M, we get $\varphi > \rho$ and $A\varphi > \rho$ on M. □

It remains to prove Proposition 1.2.

LEMMA 1.6. *Each point $p \in M$ has a relatively compact connected neighborhood V such that, for each point $q \in V$, there is a C^∞ nonnegative function ρ on M such that $\rho \equiv 0$ on $M \setminus V$, $\rho > 0$ on V, and q is the unique critical point of ρ in V (hence $\rho(q) = \max \rho$).*

Proof. We may assume without loss of generality that M is an open subset of $\mathbf{R}^n$, p is in the cube $V = (-1,1) \times \cdots \times (-1,1)$, and $V \subset\subset M$. Let $q = (a_1, \dots, a_n) \in V$ and, for each $i = 1, \dots, n$, fix a C^∞ function $\lambda_i \colon \mathbf{R} \to [0, \infty)$ such that $\lambda_i \equiv 0$ on $\mathbf{R} \setminus (-1,1)$, $\lambda_i > 0$ on $(-1,1)$, and a_i is the unique critical point of λ_i in $(-1,1)$; for example, the function

$$\lambda_i(t) = \begin{cases} \exp\left(-\dfrac{(t-a_i)^2}{1-t^2}\right) & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geq 1. \end{cases}$$

The function ρ given by

$$\rho(x) = \prod_{i=1}^{n} \lambda_i(x_i) \qquad \forall x = (x_1, \dots, x_n) \in M$$

then has the required properties. $\square$

LEMMA 1.7. *Let $\varphi \colon M \to (r,s) \subset \mathbf{R}$ be a C^2 function, let K be a compact subset of M which does not contain any critical points for φ, and let $\chi \colon (r,s) \to \mathbf{R}$ be a C^2 function satisfying $\chi'' \geq |\chi'|$ and $\chi'' \geq |\chi|$. Then, for every ϵ and R with $1 \gg \epsilon > 0$ and $R \gg 0$, we have $A[\chi(R\varphi)] \geq \epsilon R^2 \chi''(R\varphi)$ on K.*

Proof. Locally, we have

$$A = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$$

(with $a_{ij} = a_{ji}$). Hence, for $R > 0$, we have

$$\begin{aligned} A[\chi(R\varphi)] &= \sum a_{ij} \chi'(R\varphi) R \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum a_{ij} \chi''(R\varphi) R^2 \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \\ &\qquad + \sum b_i \chi'(R\varphi) R \frac{\partial \varphi}{\partial x_i} + c\chi(R\varphi) \\ &= R^2 \chi''(R\varphi) \sum a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \\ &\qquad + R\chi'(R\varphi) \left[\sum a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum b_i \frac{\partial \varphi}{\partial x_i} \right] + c\chi(R\varphi). \end{aligned}$$

Since A is elliptic with continuous coefficients and $d\varphi \neq 0$ at each point in the compact set K, it follows that there exist constants $\delta > 0$ and $N > 0$ (which do not depend on R) such that, at each point in K,

$$A[\chi(R\varphi)] \geq R^2\delta\chi''(R\varphi) - R|\chi'(R\varphi)|N - N|\chi(R\varphi)| \geq \chi''(R\varphi)(\delta R^2 - RN - N).$$

We have $\delta R^2 - RN - N > \epsilon R^2$ for $\frac{1}{2}\delta > \epsilon > 0$ and $R \gg 0$, so the claim follows. □

LEMMA 1.8. *There exists a* C^∞ *function* $\chi\colon \mathbf{R} \to \mathbf{R}$ *such that*

(i) $\chi(t) = 0$ *for* $t \leq 0$, *and*

(ii) $\chi''(t) \geq \chi'(t) \geq \chi(t) > 0$ *for* $t > 0$.

Proof. For example, if $a, b \geq 1$, then the C^∞ function

$$\chi(t) = \begin{cases} \exp\bigl(at - (b/t)\bigr) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

satisfies (i) and (ii). □

Lemmas 1.6–1.8 allow one to produce bump functions which are subsolutions outside a small set. To push the bad set off to infinity, we require chains of such bump functions. For this, we recall an elementary fact from point set topology. It is convenient and instructive to have this fact in a form which is slightly stronger than is needed at present.

LEMMA 1.9. *Let* X *be a connected, locally connected, locally compact, Hausdorff topological space, let* $\mathcal{B}$ *be a countable collection of connected open subsets which is a basis for the topology in* X, *and let* U *be a connected open subset which is not relatively compact in* X. *Suppose that there exists a connected noncompact closed subset of* X *which is contained in* U. *Then*

(i) *for any connected noncompact closed subset* C *of* X *with* $C \subset U$, *there exists a sequence of connected open subsets* $\{U_\nu\}$ *of* X *such that* $C \subset U_1$, $U = \bigcup_{\nu=1}^{\infty} U_\nu$, *and, for each* ν, $\overline{U}_\nu$ *is noncompact and* $\overline{U}_\nu \subset U_{\nu+1}$; *and*

(ii) *for each point* $p \in U$, *there exists a sequence of basis elements* $\{B_j\}$ *which tend to infinity (i.e.* $\{B_j\}$ *is a locally finite family in* X) *such that* $p \in B_1$ *and, for each* j, $B_j \subset\subset U$ *and* $B_j \cap B_{j+1} \neq \varnothing$.

If, in addition, X *is locally path connected, then*

(iii) *for each point* $p \in U$, *there is a proper continuous map* $\gamma\colon [0, \infty) \to X$ *with* $\gamma(0) = p$ *and* $\gamma([0, \infty)) \subset U$ *(i.e. a path in* U *from* p *to* ∞).

REMARK. Conversely, each of the properties (ii) and (iii) clearly implies the existence of a connected noncompact closed subset of X which is contained in U.

Proof. We first observe that there is a sequence of connected open subsets $\{\Omega_\nu\}$ of U such that $U = \bigcup_{\nu=1}^{\infty} \Omega_\nu$ and, for each ν, $\Omega_\nu \subset\subset \Omega_{\nu+1}$. For we may choose a covering of U by a sequence of basis elements $\{G_j\}$ which are relatively compact in U. For each ν, let Γ_ν be the connected component of $G_1 \cup \cdots \cup G_\nu$ containing G_1. Then $\Gamma \equiv \bigcup \Gamma_\nu$ is equal to U. For if $p \in \overline{\Gamma} \cap U$, then $p \in G_j$ for some j and G_j must meet Γ_μ for some μ. Therefore, for $\nu > \max(j, \mu)$, $G_j \cup \Gamma_\mu$ is a connected subset of $G_1 \cup \cdots \cup G_\nu$ containing G_1 and hence

$$p \in G_j \subset \Gamma_\nu \subset \Gamma.$$

Thus Γ is both open and closed relative to U and is therefore equal to U. A suitable subsequence of $\{\Gamma_\nu\}$ (chosen inductively) will then have each term relatively compact in the next term, as required. We may also choose a sequence of open subsets $\{\Theta_\nu\}$ such that $X = \bigcup_{\nu=1}^{\infty} \Theta_\nu$ and, for each ν, $\Theta_\nu \subset\subset \Theta_{\nu+1}$. Set $\Omega_0 = \Theta_0 = \Theta_{-1} = \varnothing$.

Next, we observe that, for any set C as in (i), there is a countable locally finite (in X) covering $\mathcal{A}_C$ of C by basis elements which are relatively compact in U. For we may take $\mathcal{A}_C = \bigcup_{\nu=1}^{\infty} \mathcal{A}_C^{(\nu)}$, where, for each $\nu = 1, 2, 3, \ldots$, $\mathcal{A}_C^{(\nu)}$ is a finite covering of the compact set $C \cap (\overline{\Theta}_\nu \setminus \Theta_{\nu-1})$ by basis elements which are relatively compact in $U \setminus \overline{\Theta}_{\nu-2}$.

For the proof of (i), we may choose the sequence $\{\Omega_\nu\}$ so that $\Omega_1 \cap C \neq \varnothing$. Let $C_0 = C$ and $\Omega_0 = U_0 = \varnothing$. Given connected open sets $U_0, \ldots, U_\nu$ and connected closed sets $C_0, \ldots, C_\nu$ such that, for $\mu = 1, \ldots, \nu$, we have

$$\overline{\Omega}_{\mu-1} \cup C_{\mu-1} \subset U_\mu \subset \overline{U}_\mu = C_\mu \subset U$$

(which holds vacuously if $\nu = 0$), we may choose $U_{\nu+1}$ to be the union of those elements of the collection $\mathcal{A}_{\overline{\Omega}_\nu \cup C_\nu}$ which meet the connected noncompact closed set $\overline{\Omega}_\nu \cup C_\nu$ and set $C_{\nu+1} = \overline{U}_{\nu+1}$. Proceeding, we get a sequence $\{U_\nu\}$ with the required properties.

For the proof of (ii), we may fix a connected noncompact closed subset C of X with $p \in C \subset U$ (for this, we may take $\{U_\nu\}$ as in (i) and let $C = \overline{U}_\nu$ for some $\nu \gg 0$). For each point $q \in C$, there is a finite sequence of elements $B_1, \ldots, B_k$ of $\mathcal{A}_C$ which forms a *chain from p to q*; that is, $p \in B_1$, $q \in B_k$, and $B_j \cap B_{j+1} \neq \varnothing$ for $j = 1, \ldots, k-1$ (we will call k the *length* of the chain). For the set E of points q in C for which there is a chain from p to q is clearly nonempty and open relative to C. On the other hand, E is also closed because, if $q \in \overline{E}$, then $q \in B$ for some set $B \in \mathcal{A}_C$ and there must be some point $r \in B \cap E$. A chain $B_1, \ldots, B_k$ from p to r yields the chain $B_1, \ldots, B_k, B$ from p to q. Thus $E = C$. Observe that if

$q \in E$ and $B_1, \ldots, B_k$ is a chain of minimal length from p to q, then the sets $B_1, \ldots, B_k$ are distinct.

Now since C is noncompact and closed, we may choose a sequence of points $\{q_\nu\}$ in C with $q_\nu \to \infty$ in X (for example, $q_\nu \in C \setminus \Theta_\nu$ for each ν) and, for each ν, we may choose a chain $B_1^{(\nu)}, \ldots, B_{k_\nu}^{(\nu)}$ of minimal length from p to q_ν. Since the elements of $\mathcal{A}_C$ are relatively compact in U and $\mathcal{A}_C$ is locally finite in X, there are only finitely many possible choices for $B_j^{(\nu)}$ for each j (only finitely many elements of $\mathcal{A}_C$ will be in some chain of length j from p). Moreover, for each fixed $j \in \mathbf{N}$, we have $k_\nu > j$ for $\nu \gg 0$, because the set of points in C joined to p by a chain of length $\leq j$ is relatively compact in C while $q_\nu \to \infty$. Therefore, after applying a diagonal argument and passing to the associated subsequence of $\{q_\nu\}$, we may assume that, for each j, there is an element $B_j \in \mathcal{A}_C$ with $B_j^{(\nu)} = B_j$ for all $\nu \gg 0$. Thus we get an infinite chain of distinct elements $\{B_j\}$ from p to infinity as required in (ii) (local finiteness in X is guaranteed since $\mathcal{A}_C$ is locally finite and the elements $\{B_j\}$ are distinct).

Finally, suppose X is locally path connected. Then the sets $\{B_j\}$ as in (ii) are path connected and, setting $p_0 = p$ and, choosing $p_j \in B_j \cap B_{j+1}$ for each $j = 1, 2, 3, \ldots$, we may take $\gamma|_{[j-1,j]}$ to be a path in B_j from p_{j-1} to p_j for each j. $\square$

Proof of Proposition 1.2. We first observe that, if V is a set with the properties described in Lemma 1.6, D is a compact subset of V, and W is a nonempty open subset of $V \setminus D$, then there is a nonnegative C^∞ function β with compact support in V such that $A\beta \geq 0$ on $M \setminus W$ and $\beta > 0$ and $A\beta > 1$ on D. For we may choose a point $q \in W$, a C^∞ nonnegative function ρ on M which is positive on V and has unique critical point q in V, a C^∞ function χ on $\mathbf{R}$ as in Lemma 1.8, and a constant $\epsilon > 0$ with $\rho > \epsilon$ on D. By Lemma 1.7 (applied to the compact set $K = \{x \in M \setminus W \mid \rho(x) \geq \epsilon\} \subset V \setminus W$), for $R \gg 0$, the function $\beta \equiv \chi(R(\rho - \epsilon))$ will have the required properties.

Next, by Lemma 1.9, given a point $p \in U$, there is a locally finite (in X) sequence of relatively compact open subsets $\{V_m\}$ of U such that $p \in V_1$ and, for each m, V_m has the properties described in Lemma 1.6 and $V_m \cap V_{m+1} \neq \varnothing$. Hence we may choose a sequence of disjoint nonempty open sets $\{W_m\}_{m=0}^{\infty}$ such that $p \in W_0 \subset\subset V_1$ and, for each $m \geq 1$, $W_m \subset\subset V_m \cap V_{m+1}$.

By the first observation, there is a sequence of nonnegative C^∞ functions $\{\beta_m\}_{m=1}^{\infty}$ such that, for each m, β_m is compactly supported in V_m, $A\beta_m \geq 0$

on $M \setminus W_m$, and $\beta_m > 0$ and $A\beta_m > 1$ on $\overline{W}_{m-1}$. We will choose positive constants $\{R_m\}$ inductively so that, for each $m = 1, 2, 3, \ldots$,

$$A\left(\sum_{j=1}^{m} R_j\beta_j\right) \begin{cases} \geq 0 & \text{on } M \setminus W_m , \\ > 1 & \text{on } \overline{W}_0 . \end{cases}$$

Let $R_1 \geq 1$. Given $R_1, \ldots, R_{m-1} > 0$ with the above property, using the fact that $A\beta_m > 1$ on $\overline{W}_{m-1}$, we get, for $R_m \gg 0$,

$$A\left(\sum_{j=1}^{m} R_j\beta_j\right) > 1 \quad \text{on } \overline{W}_{m-1} .$$

On $M \setminus (W_{m-1} \cup W_m)$ we have $A\beta_m \geq 0$ and hence

$$A\left(\sum_{j=1}^{m} R_j\beta_j\right) \geq A\left(\sum_{j=1}^{m-1} R_j\beta_j\right) \geq 0 .$$

On $\overline{W}_0$, the above middle expression, and hence the expression on the left, is greater than 1. Proceeding, we get the sequence $\{R_m\}$. The sum $\sum R_m\beta_m$ is locally finite in X and the sequence of sets $\{W_m\}$ is locally finite in X, so the sum converges to a function α with the required properties. □

A slight modification of the proof of Theorem 1.1 gives the following more general version:

THEOREM 1.10. *Suppose K is a compact subset of M whose complement $M \setminus K$ has no relatively compact components, ρ is a positive continuous function on M, and W is a neighborhood of K in M. Then there exists a C^∞ function φ on M such that*

(i) *$\varphi \geq 0$ and $A\varphi \geq 0$ on M,*

(ii) *$\varphi > \rho$ and $A\varphi > \rho$ on $M \setminus W$,*

(iii) *$\varphi \equiv 0$ on K, and*

(iv) *$\varphi > 0$ and $A\varphi > 0$ on $M \setminus K$.*

Proof. We proceed as in the proof of Theorem 1.1 but now with $K_0 = K$. By Lemma 1.5, we may choose nonempty compact sets $\{K_\nu\}$ such that $M = \bigcup_{\nu=1}^{\infty} K_\nu$ and such that, for each $\nu = 0, 1, 2, \ldots$, we have $K_\nu \subset \mathring{K}_{\nu+1}$ and $M \setminus K_\nu$ has no relatively compact components.

Given a point $p \in M \setminus K$, there is a unique $\nu = \nu(p) \geq 0$ with $p \in K_{\nu+1} \setminus K_\nu$ and, by Proposition 1.3, there is a C^∞ nonnegative function α_p and a relatively compact neighborhood V_p of p in $M \setminus K_\nu$ such that $A\alpha_p \geq 0$ on M, $\alpha_p \equiv 0$ on K_ν, and $\alpha_p > \rho$ and $A\alpha_p > \rho$ on V_p. Thus we may choose a sequence of points $\{p_k\}$ in $M \setminus K$ and corresponding functions $\{\alpha_{p_k}\}$ and neighborhoods $\{V_{p_k}\}$ so that $\{V_{p_k}\}$ forms a covering of $M \setminus W$ which is locally finite in M (as in the proof of Theorem 1.1, one may take $\{p_k\}$ to be an enumeration of $\bigcup_{\nu=0}^{\infty} Z_\nu$ where, for each ν, Z_ν is a finite set of points in $M \setminus (W \cup \overset{\circ}{K}_\nu)$ such that $\{V_p\}_{p \in Z_\nu}$ covers $K_{\nu+1} \setminus (W \cup \overset{\circ}{K}_\nu)$). The collection $\{\operatorname{supp} \alpha_{p_k}\}$ is then locally finite in M and the locally finite sum $\sum_{k=1}^{\infty} \alpha_{p_k}$ converges to a C^∞ function ψ on M satisfying $\psi \geq \alpha_{p_k}$ and $A\psi \geq A\alpha_{p_k}$ for each k. It follows that $\psi \geq 0$ and $A\psi \geq 0$ on M, $\psi \equiv 0$ on $K = K_0$, and $\psi > \rho$ and $A\psi > \rho$ on $M \setminus W$.

To obtain the properties (iv), we choose a sequence of points $\{q_m\}$ in $M \setminus K$ and corresponding functions $\{\alpha_{q_m}\}$ and neighborhoods $\{V_{q_m}\}$ so that $\{V_{q_m}\}$ covers $W \setminus K$. Applying a diagonal argument, we may choose a sequence of positive numbers $\{\epsilon_m\}$ converging to 0 so fast that each derivative of arbitrary order for the sequence of partial sums of $\sum \epsilon_m \alpha_{q_m}$ converges uniformly on compact subsets of M. The function $\varphi = \psi + \sum \epsilon_m \alpha_{q_m}$ will then have the required properties. □

The main topological fact required in the proof of Proposition 1.2 was part (ii) of Lemma 1.9. This fact is slightly easier to verify for U a component of the complement of a compact set. But the more general version (as stated in Lemma 1.9) and the proof of Proposition 1.2 actually yield the following:

PROPOSITION 1.11. *Let U be a connected open subset of M which contains a connected noncompact closed subset of M. Then, for each point $p \in U$, there exists a C^∞ function α such that*

(i) $\alpha \geq 0$ *and* $A\alpha \geq 0$ *on* M,

(ii) $\operatorname{supp} \alpha \subset U$,

(iii) $\alpha(p) > 0$, *and*

(iv) $A\alpha(p) > 0$.

PROPOSITION 1.12. *Let K be a closed subset of M such that each component of $M \setminus K$ contains a connected noncompact closed subset of M. Then, for each point $p \in M \setminus K$, there is a C^∞ nonnegative function α on M such that $A\alpha \geq 0$ on M, $\alpha \equiv 0$ on K, $\alpha(p) > 0$, and $A\alpha(p) > 0$.*

We also get a corresponding generalization of Theorem 1.10:

THEOREM 1.13. *Suppose* K *is a closed subset of* M *such that each component of* $M \setminus K$ *contains a connected noncompact closed subset of* M *and* $D \subset M \setminus K$ *is a closed subset of* M *with no compact components. Then, for every positive continuous real-valued function* ρ *on* M, *there is a* C^∞ *function* φ *such that*

(i) $\varphi \geq 0$ *and* $A\varphi \geq 0$ *on* M,
(ii) $\varphi > \rho$ *and* $A\varphi > \rho$ *on* D,
(iii) $\varphi \equiv 0$ *on* K, *and*
(iv) $\varphi > 0$ *and* $A\varphi > 0$ *on* $M \setminus K$.

Before addressing the proof, we consider some remarks.

REMARKS. 1. If the coefficients of A are (for example) C^1 and the constant term is nonpositive, then the existence of a connected noncompact closed subset C of M with $C \subset U$ is necessary in Proposition 1.11. In fact, if U is a connected open subset of M and M admits a nonconstant nonnegative upper semi-continuous subsolution α which vanishes on $M \setminus U$, then U must contain such a set. For $\alpha(p) > 0$ at some point $p \in U$ and hence we may choose a number ϵ with $0 < \epsilon < \alpha(p)$ and a neighborhood V of the closed set $\{x \in M \mid \alpha(x) \geq \epsilon\}$ with $\overline{V} \subset U$. The maximum principle then implies that the component W of V containing p is not relatively compact in M. Thus the set $C = \overline{W}$ is a closed connected noncompact subset of M contained in U.

In particular, as the following example illustrates, the conclusions of Proposition 1.2 and Proposition 1.3 do *not* hold in general for K a closed noncompact set.

EXAMPLE 1.14. Let K be the closed subset of the manifold $M = \mathbf{R}^2 \setminus \{(0,0)\}$ given by

$$K = \big(M \setminus (0,2) \times (0,2)\big) \cup \bigcup_{m=1}^{\infty} \{1/m\} \times [0,1]\,.$$

Then the complement $U = M \setminus K$ is connected and $\overline{U}$ is noncompact. But U does not contain a connected noncompact closed subset of M and, therefore, every nonnegative upper semi-continuous subharmonic function φ on M which vanishes on K must vanish everywhere in M.

2. As the following example shows, the conclusion of Theorem 1.13 may fail to hold if the set $D \subset M \setminus K$ has compact components.

EXAMPLE 1.15. The complement $U = M \setminus K$ in $M = \mathbf{R}^2 \setminus \{(0,1)\}$ of the closed set

$$K = \big(M \setminus (0,\infty) \times (0,4)\big) \cup \bigcup_{m=1}^{\infty} \{1/(2m)\} \times [0,2]$$

is a connected set with noncompact closure and U contains the closed noncompact connected set $C = [1,\infty) \times \{3\}$. The noncompact subset

$$D = \{\, (1/(2m+1), 1) \mid m \in \mathbf{N} \,\}$$

of U is closed (in fact, discrete) in M. If φ is a nonnegative upper semi-continuous subharmonic function on M which vanishes on K, then, for each m, applying the maximum principle in $[1/(2m+2), 1/(2m)] \times [0,2]$, we get a number $\{r_m\}$ with $1/(2m+2) < r_m < 1/(2m)$ and $\varphi(r_m, 2) \geq \varphi(1/(2m+1), 1)$. Since $(r_m, 2) \to (0,2) \in K \subset M$ and φ is upper semi-continuous, it follows that φ must be bounded on D.

3. If $K \subset M$ is a compact set, then one can achieve the conditions in Proposition 1.3 and Theorem 1.10 by replacing K by the compact set $\widehat{K}$. Because we have Proposition 1.12 and Theorem 1.13, for a general closed set $K \subset M$ it is natural to define $\widehat{K}$ to be the union of K with all components of $M \setminus K$ which do *not* contain any connected noncompact closed subsets of M.

The main step in the proof of Theorem 1.13 is the case in which D and $M \setminus K$ are connected.

LEMMA 1.16. *Suppose U is a connected open subset of M, C is a connected noncompact closed subset of M with $C \subset U$, and ρ is a positive continuous function on M. Then there is a C^∞ function φ such that*

(i) *$\varphi \geq 0$ and $A\varphi \geq 0$ on M,*
(ii) *$\varphi > \rho$ and $A\varphi > \rho$ on C,*
(iii) *$\varphi \equiv 0$ on $M \setminus U$, and*
(iv) *$\varphi > 0$ and $A\varphi > 0$ on U.*

Proof. We first show that there is a nonnegative C^∞ function ψ such that $A\psi \geq 0$ on M, $\psi \equiv 0$ on $M \setminus U$, and $\psi > \rho$ and $A\psi > \rho$ on C.

For this purpose, we may assume without loss of generality that C is locally connected. For we may choose (as in the proof of Lemma 1.9) a locally finite (in M) covering $\mathcal{A}$ of C by relatively compact connected open subsets of U. We may also choose the covering so that each element meets C and has locally connected closure (for example, we may choose $\mathcal{A}$ so that, for each $B \in \mathcal{A}$, there is a diffeomorphism of some neighborhood of $\bar{B}$ onto an open subset of $\mathbf{R}^n$ mapping B onto a ball). The closed connected noncompact set

$$C' \equiv \overline{\bigcup_{B\in\mathcal{A}} B} = \bigcup_{B\in\mathcal{A}} \bar{B} \subset U$$

is then locally connected. For if $p \in C'$ and $B_1, \dots, B_k$ are the (finitely many) elements of $\mathcal{A}$ whose closures contain p, then, for each $j = 1, \dots, k$, we may choose a neighborhood W_j of p in M such that $W_j \cap \bar{B}_j$ is connected and $W_j \cap \bar{B} = \varnothing$ for each set $B \in \mathcal{A} \setminus \{B_1, \dots, B_k\}$. The set $D \equiv \bigcup_{j=1}^k (W_j \cap \bar{B}_j)$ is then a connected subset of C' which contains the set $W_1 \cap \cdots \cap W_k \cap C'$, a neighborhood of p relative to C'. It follows that C' is locally connected (since, by choosing the neighborhoods $\{W_j\}$ small, one sees that the components of any open subset of C' are open relative to C'). Therefore, by replacing C with the set C', we may assume that C is locally connected.

By Lemma 1.4, there is a sequence of compact sets $\{K_\nu\}$ such that $M = \bigcup_{\nu=1}^\infty K_\nu$ and, for each ν, we have $K_\nu \subset \mathring{K}_{\nu+1}$ and $C \setminus K_\nu$ has only finitely many components, all of which have noncompact closure. For we may choose inductively a sequence of compact subsets $\{K'_\nu\}$ of M such that $M = \bigcup_{\nu=1}^\infty K'_\nu$ and such that, for each ν, we have

$$K_\nu \equiv K'_\nu \cup (\widehat{K'_\nu \cap C})_C \subset \mathring{K}'_{\nu+1}\,;$$

where, for $K \subset C$ compact, $\widehat{K}_C$ is the union of K with all of the relatively compact components of $C \setminus K$.

We now proceed as in the proofs of Theorem 1.1 and Theorem 1.10. Let $K_0 = \varnothing$. Given $p \in C$, there is a unique $\nu = \nu(p)$ with $p \in K_{\nu+1} \setminus K_\nu$. The component U_p of $U \setminus K_\nu$ containing p must also contain the closure of some component of $C \setminus K_{\nu+1} \subset C \setminus \mathring{K}_{\nu+1} \subset C \setminus K_\nu$. For we may take a point q in the component of $C \setminus K_\nu$ containing p (a set with noncompact closure) which lies outside $K_{\nu+1}$. The closure of the component of $C \setminus K_{\nu+1}$ containing q is then contained in U_p. We may apply Proposition 1.11 to get a C^∞ nonnegative function α_p and a relatively compact neighborhood V_p of p in U_p such that $A\alpha_p \geq 0$ on M, $\operatorname{supp} \alpha_p \subset U_p$, and $\alpha_p > \rho$ and $A\alpha_p > \rho$ on V_p. Thus we may choose a sequence of points $\{p_k\}$ in C and corresponding functions $\{\alpha_{p_k}\}$ and neighborhoods $\{V_{p_k}\}$ so that $\{V_{p_k}\}$ forms a locally finite (in M) covering of C. The collection $\{\operatorname{supp} \alpha_{p_k}\}$ is then locally finite in M

because $\operatorname{supp}\alpha_{p_k} \subset U \setminus K_\nu$ whenever $p_k \notin K_\nu$. Hence the sum $\sum_{k=1}^\infty \alpha_{p_k}$ is locally finite and, therefore, convergent to a C^∞ function ψ on M with the required properties.

Now, by Lemma 1.9, there is a sequence of connected open sets $\{U_\nu\}$ such that $U = \bigcup_{\nu=1}^\infty U_\nu$, $C \subset U_1$, and, for each ν, $\overline{U}_\nu \subset U_{\nu+1}$. By the above, we may form a C^∞ nonnegative function ψ such that $A\psi \geq 0$ on M, $\psi \equiv 0$ on $M \setminus U_1$, and $\psi > \rho$ and $A\psi > \rho$ on C and, for each $\nu = 1, 2, 3, \ldots$, we may form a C^∞ nonnegative function ψ_ν such that $A\psi_\nu \geq 0$ on M, $\psi_\nu \equiv 0$ on $M \setminus U_{\nu+1}$, and $\psi_\nu > 1$ and $A\psi_\nu > 1$ on $\overline{U}_\nu$. Choosing a sequence of positive numbers $\{\epsilon_\nu\}$ converging to 0 sufficiently fast, the function $\varphi = \psi + \sum \epsilon_\nu \psi_\nu$ will have the required properties. □

For the general case, we will apply the following:

LEMMA 1.17. *Suppose X is a second countable, connected, locally connected, locally compact, Hausdorff topological space; K is a closed subset of X; and $D \subset X \setminus K$ is a closed subset of X with no compact components. Then there exists a countable locally finite (in X) family of disjoint connected noncompact closed sets $\{C_\lambda\}_{\lambda\in\Lambda}$ and a locally finite (in X) family of disjoint connected open sets $\{U_\lambda\}_{\lambda\in\Lambda}$ such that*

$$D \subset C \equiv \bigcup_{\lambda\in\Lambda} C_\lambda \quad \text{and} \quad C_\lambda \subset U_\lambda \subset \overline{U}_\lambda \subset X \setminus K \quad \forall \lambda \in \Lambda.$$

REMARK. We will not use the fact that the sets $\{U_\lambda\}$ are disjoint.

Proof. As in the proof of Lemma 1.9, there is a countable locally finite covering $\mathcal{A}_D$ of D by connected open relatively compact subsets of $X \setminus K$ which meet D. Thus

$$D \subset V \equiv \bigcup_{B\in\mathcal{A}_D} B \subset \overline{V} = \bigcup_{B\in\mathcal{A}_D} \overline{B} \subset X \setminus K$$

(where we have used the local finiteness of the collection $\mathcal{A}_D$). Since each of the components of V meets, and therefore contains, a component of D, the family of components $\{V_\gamma\}_{\gamma\in\Gamma}$ of V is a locally finite family of connected open sets with noncompact closure. The set

$$C \equiv \overline{V} = \bigcup_{\gamma\in\Gamma} \overline{V}_\gamma$$

is a closed set contained in $X \setminus K$ and the family of components $\{C_\lambda\}_{\lambda\in\Lambda}$ of C satisfies

$$C_\lambda = \bigcup_{\substack{\gamma\in\Gamma \\ \bar{V}_\gamma \subset C_\lambda}} \bar{V}_\gamma = \overline{\bigcup_{\substack{\gamma\in\Gamma \\ \bar{V}_\gamma \subset C_\lambda}} V_\gamma} \quad \forall \lambda \in \Lambda .$$

It follows that the family is locally finite in X (since the family $\{\bar{V}_\gamma\}_{\gamma\in\Gamma}$ is locally finite) and that C_λ is closed for each $\lambda \in \Lambda$. Consequently, we may choose a locally finite covering $\mathcal{A}_C$ of C by connected relatively compact open subsets of $X \setminus K$ such that, for each element $B \in \mathcal{A}_C$, B and $\bar{B}$ meet exactly one component of C. For each $\lambda \in \Lambda$, taking U_λ to be the component of the set

$$\bigcup_{\substack{B\in\mathcal{A}_C \\ B\cap C_\lambda \neq \varnothing}} B \setminus \bigcup_{\substack{B\in\mathcal{A}_C \\ B\cap C_\lambda = \varnothing}} \bar{B}$$

containing C_λ, we get disjoint connected open sets $\{U_\lambda\}_{\lambda\in\Lambda}$ with

$$C_\lambda \subset U_\lambda \subset \bar{U}_\lambda \subset X \setminus K$$

for each $\lambda \in \Lambda$. This family is locally finite in X. For each point in X has a neighborhood Q which meets only finitely many elements $B_1, \ldots, B_k$ of $\mathcal{A}_C$. Each B_j meets a unique component C_{λ_j} of C. If $\lambda \in \Lambda$ with $Q \cap U_\lambda \neq \varnothing$, then $Q \cap B \neq \varnothing$ for some $B \in \mathcal{A}_C$ with $B \cap C_\lambda \neq \varnothing$. Hence we must have $B = B_j$ for some j and, therefore, $\lambda = \lambda_j$. □

Proof of Theorem 1.13. Let

$$D \subset C \equiv \bigcup_{\lambda\in\Lambda} C_\lambda \qquad \text{and} \qquad C_\lambda \subset U_\lambda \subset \bar{U}_\lambda \subset M \setminus K \quad \forall \lambda \in \Lambda$$

be as in Lemma 1.17. Applying Lemma 1.16 to each pair of sets $C_\lambda \subset U_\lambda$, we get a nonnegative C^∞ function α_λ such that $A\alpha_\lambda \geq 0$ on M, $\alpha_\lambda \equiv 0$ on $M \setminus U_\lambda$, and $\alpha_\lambda > \rho$ and $A\alpha_\lambda > \rho$ on C_λ (we do not need the properties (iv) of Lemma 1.16 for this part). Since the family $\{U_\lambda\}$ is locally finite in M, the sum $\sum \alpha_\lambda$ determines a nonnegative C^∞ function α with $A\alpha \geq 0$ on M, $\alpha \equiv 0$ on $M \setminus \bigcup_{\lambda\in\Lambda} U_\lambda \supset K$, and $\alpha > \rho$ and $A\alpha > \rho$ on $C \supset D$.

Applying Lemma 1.16 to each of the components $\{V_j\}_{j\in J}$ of $M \setminus K$, we get, for each $j \in J$, a C^∞ nonnegative function β_j such that $A\beta_j \geq 0$ on M, $\beta_j \equiv 0$ on $M \setminus V_j$, and $\beta_j > 0$ and $A\beta_j > 0$ on V_j. For J a finite set, we may now take $\varphi = \alpha + \sum_{j\in J} \beta_j$. If J is infinite, then, assuming as we may that $J = \mathbf{N}$ and choosing a sequence of positive numbers $\{\epsilon_j\}$ converging to 0 sufficiently fast, the function $\varphi = \alpha + \sum_{j=1}^\infty \epsilon_j \beta_j$ will have the required properties. □

We close this section with the following observation concerning Theorem 1.10 for K the closure of a smooth relatively compact domain.

COROLLARY 1.18. *Suppose Ω is a C^∞ relatively compact domain in M whose complement $M \setminus \Omega$ has no compact components, ρ is a positive continuous function on M, and W is a neighborhood of $\overline{\Omega}$. Then there is a C^∞ function φ on M such that $\varphi > \rho$ on $M \setminus W$, $A\varphi > \rho$ on M, 0 is a regular value for φ, and $\Omega = \{ x \in M \mid \varphi(x) < 0 \}$.*

Proof. There exists a C^∞ function τ on M such that 0 is a regular value for τ, τ is locally constant on $M \setminus V$ for some relatively compact neighborhood V of $\partial\Omega$ in W, and $\Omega = \{ x \in M \mid \tau(x) < 0 \}$. For $\epsilon > 0$ sufficiently small, we have

$$D \equiv \{ x \in M \mid -2\epsilon \leq \tau(x) \leq 2\epsilon \} \subset V$$

and $(d\tau)_x \neq 0$ for each point $x \in D$. By Theorem 1.1, there is a C^∞ function α with compact support in Ω such that $\alpha \leq 0$ on M and $A\alpha > \rho$ on $\{ x \in M \mid \tau(x) \leq -\epsilon \}$. By Theorem 1.10, there is a C^∞ nonnegative function β on M such that $A\beta \geq 0$ on M, $\beta \equiv 0$ on Ω, $\beta > 0$ and $A\beta > 0$ on $M \setminus \overline{\Omega}$, and $\beta > \rho$ and $A\beta > 1 + \rho$ on $\{ x \in M \mid \tau(x) \geq \epsilon \}$. Finally, we may fix a C^∞ function $\chi \colon \mathbf{R} \to [0, \infty)$ as in Lemma 1.8. Let $R_1, R_2, R_3 > 1$ and let

$$\varphi = \alpha + R_2 \chi\big(R_1(\tau + 2\epsilon)\big) - R_2 \chi(2R_1\epsilon) + R_3 \beta .$$

On $M \setminus W$, we have $\varphi \geq R_3 \beta > \beta > \rho$. On Ω, we have

$$\varphi < 0 + R_2 \chi\big(R_1(0 + 2\epsilon)\big) - R_2 \chi(2R_1\epsilon) + R_3 \cdot 0 = 0 .$$

On $M \setminus \overline{\Omega}$, we have

$$\varphi \geq 0 + R_2 \chi\big(R_1(0 + 2\epsilon)\big) - R_2 \chi(2R_1\epsilon) + R_3 \beta \geq R_3 \beta > 0 .$$

Thus $\Omega = \{ x \in M \mid \varphi(x) < 0 \}$. For any point $x \in \partial\Omega = \{ x \in M \mid \varphi(x) = 0 \}$, we have $\alpha = 0$ near x and β has a local minimum at x. Thus

$$d\varphi = d\alpha + R_1 R_2 \chi'(2R_1\epsilon) d\tau + R_3 d\beta = 0 + R_1 R_2 \chi'(2R_1\epsilon) d\tau + 0 \neq 0 .$$

By Lemma 1.7, for $R_1 \gg 0$, we get $A\big[\chi\big(R_1(\tau + 2\epsilon)\big)\big] \geq 0$ on $\Omega \cup D = \{ x \in M \mid \tau(x) \leq 2\epsilon \}$ and $A\big[\chi\big(R_1(\tau + 2\epsilon)\big)\big] > 0$ on $\{ x \in M \mid -2\epsilon < \tau(x) \leq 2\epsilon \} \subset D$. On $\{ x \in M \mid \tau(x) \leq -\epsilon \}$ we have

$$A\varphi \geq A\alpha + 0 + 0 = A\alpha > \rho .$$

For $R_2 \gg 0$, on $\{ x \in M \mid -\epsilon \leq \tau(x) \leq \epsilon \}$ we have

$$A\varphi \geq A\alpha + R_2 A\big[\chi\big(R_1(\tau + 2\epsilon)\big)\big] > \rho .$$

Finally, since τ is locally constant on $M \setminus V$, for $R_3 \gg 0$, on $\{ x \in M \mid \tau(x) \geq \epsilon \} \subset M \setminus \overline{\Omega}$ we have

$$A\varphi = 0 + R_2 A\big[\chi\big(R_1(\tau+2\epsilon)\big)\big] + R_3 A\beta > R_2 A\big[\chi\big(R_1(\tau+2\epsilon)\big)\big] + R_3(1+\rho) > \rho . \quad \square$$

2. Two applications

To illustrate the broad utility of the existence of exhausting strict subsolutions, we consider two (mostly known) consequences.

We first recall that any C^k function φ on a smooth manifold can be approximated in the C^k Whitney topology by a C^∞ Morse function ψ [GoG]. Applying this to a function φ from Theorem 1.1, we get

COROLLARY 2.1. *If M and A are as in Section 1 with M noncompact and ρ is a positive continuous function on M, then there exists a C^∞ Morse function ψ satisfying $\psi > \rho$ and $A\psi > \rho$.*

Taking A to be Δ_g for a Riemannian metric g and ρ to be a continuous exhaustion function, we get a Morse exhaustion function ψ with $\Delta_g \psi > 0$. Since $\Delta_g \psi$ is the trace of the Hessian of ψ with respect to g, the Hessian has at least one positive eigenvalue at each point, so the index of ψ is at most $n - 1$. Thus we get the following well-known fact:

THEOREM 2.2. *A connected noncompact C^∞ manifold of dimension n has the homotopy type of a CW complex with cells of dimension $\leq n - 1$.*

The next observation is that the existence of exhausting strict subsolutions allows one to construct a Hermitian metric of positive scalar curvature (positive curvature in the case of a Riemann surface) in a holomorphic line bundle with a nontrivial holomorphic section.

For the rest of this section, X will denote a connected complex manifold of (complex) dimension n and g will denote a C^∞ Hermitian metric in X. The *Levi form* of a C^2 function φ on X is the Hermitian tensor given by, in local holomorphic coordinates $(z_1, \dots, z_n)$,

$$\mathcal{L}(\varphi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j .$$

The *Laplace operator* Δ_g for the Hermitian metric g is given by the trace of the Levi form:

$$\Delta_g = \sum_{i,j} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$$

where $\left(g^{i\bar{j}}\right) = \overline{\left(g_{i\bar{j}}\right)^{-1}}$. This elliptic operator is equal to $1/2$ the Laplace operator of the associated Riemannian metric if g is Kähler. A C^2 real-valued function φ is called *subharmonic* (*strictly subharmonic*) with respect to g if $\Delta_g \varphi \geq 0$ (respectively, $\Delta_g \varphi > 0$).

If L is a holomorphic line bundle on X and h is a C^2 Hermitian metric in L, then the *curvature* of h is the Hermitian tensor Θ_h given by

$$\Theta_h \equiv \mathcal{L}(-\log |s|_h^2)$$

for any nonvanishing local holomorphic section s of L. The *scalar curvature* $\mathcal{R}_h$ of h with respect to g is given by the trace of the curvature; that is, locally,

$$\mathcal{R}_h \equiv \Delta_g(-\log |s|_h^2) .$$

In particular, if X is a Riemann surface, then $\mathcal{R}_h = \Theta_h / g$.

THEOREM 2.3. *Let L be a holomorphic line bundle on X. If X is noncompact or $L = [D]$ is the holomorphic line bundle associated to a nontrivial effective divisor D in X (i.e. L is a holomorphic line bundle which admits a nontrivial global holomorphic section), then L admits a C^∞ Hermitian metric h with positive scalar curvature.*

Proof. Fix a C^∞ Hermitian metric k in L. We will modify k to obtain h.

Assuming first that X is noncompact, Theorem 1.1 provides a C^∞ strictly subharmonic (with respect to g) exhaustion function φ. If χ is a C^∞ function on $\mathbf{R}$ with $\chi' > 0$ and $\chi'' \geq 0$ and

$$h = e^{-\chi(\varphi)} k ,$$

then

$$\mathcal{R}_h = \Delta_g(\chi(\varphi)) + \mathcal{R}_k = \chi'(\varphi)\Delta_g\varphi + \chi''(\varphi)|\partial\varphi|_g^2 + \mathcal{R}_k \geq \chi'(\varphi)\Delta_g\varphi + \mathcal{R}_k\,.$$

Choosing χ so that $\chi'(t) \to \infty$ sufficiently fast as $t \to \infty$, we get $\mathcal{R}_h > 0$.

Assuming now that X is compact and $L = [D]$, where D is a nontrivial effective divisor, let $Y = |D| \subset X$ be the support of D and let s be a global holomorphic section of L with associated divisor D. Applying Theorem 1.1 to a noncompact neighborhood of Y in X and cutting off, we get a C^∞ function α on X which is strictly subharmonic on a neighborhood U of Y. After shrinking U slightly and replacing α by a large multiple, we may assume that

$$\Delta_g\alpha + \mathcal{R}_k > 2 \quad \text{on } U\,.$$

Since $-\log|s|_k^2 \to \infty$ at Y, we have, for $N \gg 0$,

$$Y \subset \{\, x \in X \mid \alpha(x) - \log|s(x)|_k^2 \geq N \,\} \subset U$$

(setting $\alpha - \log|s|_k^2 = \infty$ along Y). We may choose a C^∞ function λ on $\mathbf{R}$ such that $\lambda' \geq 0$, $\lambda'' \geq 0$, $\lambda(t) = t$ if $t \geq 3N$, and $\lambda(t) = 2N$ if $t \leq N$. We set $\lambda(\infty) = \infty$. The restriction of the function

$$\rho \equiv \lambda\big(\alpha - \log|s|_k^2\big)$$

to $X \setminus Y$ is C^∞ and subharmonic because $\rho \equiv 2N$ on a neighborhood of $X \setminus U$ (and hence $\Delta_g\rho = 0$), while on $U \setminus Y$ we have

$$\begin{aligned}\Delta_g\rho &= \lambda'\big(\alpha - \log|s|_k^2\big)\cdot\big(\Delta_g\alpha + \mathcal{R}_k\big) + \lambda''\big(\alpha - \log|s|_k^2\big)\cdot\big|\partial\big(\alpha - \log|s|_k^2\big)\big|_g^2\\ &\geq 2\lambda'\big(\alpha - \log|s|_k^2\big) \geq 0\,.\end{aligned}$$

Observe also that $\rho = \alpha - \log|s|_k^2$ on the relatively compact neighborhood V of Y in U given by

$$V = \{\, x \in X \mid \alpha(x) - \log|s(x)|_k^2 > 3N \,\}\,.$$

Applying Theorem 1.1 to the connected noncompact manifold $X \setminus Y$ and cutting off near Y, we get a C^∞ function β with compact support in $X \setminus Y$ satisfying $\Delta_g\beta > 0$ on $X \setminus V$. Choosing $\epsilon > 0$ so small that $\epsilon\Delta_g\beta > -1$ on X, we see that the restriction of the function $\gamma \equiv \rho + \epsilon\beta$ to $X \setminus Y$ is C^∞ and strictly subharmonic. In fact, on $V \setminus Y$, we have

$$\Delta_g\gamma = \Delta_g(\alpha - \log|s|_k^2 + \epsilon\beta) > 2 - 1 = 1\,.$$

We may now define $|\xi|_h^2$ for $\xi \in L_x$ with $x \in X$ by

$$|\xi|_h^2 = \begin{cases} e^{-\gamma(x)}|\xi/s(x)|^2 & \text{if } x \in X \setminus Y, \\ e^{-\alpha(x)-\epsilon\beta(x)}|\xi|_k^2 & \text{if } x \in V. \end{cases}$$

Then h is a well-defined C^∞ Hermitian metric in L since, for $x \in V \setminus Y$ and $\xi \in L_x$, we have

$$e^{-\gamma(x)}|\xi/s(x)|^2 = e^{-\left(\alpha(x)-\log|s(x)|_k^2+\epsilon\beta(x)\right)}|\xi|_k^2/|s(x)|_k^2 = e^{-\alpha(x)-\epsilon\beta(x)}|\xi|_k^2 .$$

Furthermore, on $X \setminus Y$ we have

$$\mathcal{R}_h = \Delta_g\left(-\log|s|_h^2\right) = \Delta_g\gamma \begin{cases} > 0 & \text{on } X \setminus Y \\ > 1 & \text{on } V \setminus Y \end{cases}$$

By continuity, we also have $\mathcal{R}_h \geq 1 > 0$ at points in Y. Thus $\mathcal{R}_h > 0$ on X. □

For X a Riemann surface, the above proofs become especially simple. For example, the construction of α in the proof of Theorem 2.3 is trivial for $\dim X = 1$ because Y is discrete. For X an open Riemann surface, Theorem 0.1 provides a C^∞ strictly plurisubharmonic exhaustion function and, therefore, by [Gr] and [DG], one gets the theorem of [BS] that an open Riemann surface is Stein. For a compact Riemann surface X, Theorem 2.3 becomes the familiar fact (see, for example, [GriH]) that the holomorphic line bundle associated to a nontrivial effective divisor admits a C^∞ Hermitian metric h with positive curvature Θ_h.

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