Smooth Lyapunov 1-forms

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SMOOTH LYAPUNOV 1-FORMS

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ABSTRACT. We find conditions which guarantee that a given flow Φ on a closed smooth manifold M admits a smooth Lyapunov 1-form ω lying in a prescribed de Rham cohomology class $\xi \in H^1(M;\mathbf{R})$. These conditions are formulated in terms of Schwartzman's asymptotic cycles $\mathcal{A}_\mu(\Phi) \in H_1(M;\mathbf{R})$ of the flow.

1. INTRODUCTION

C. Conley [1, 2] showed that any continuous flow $\Phi\colon X \times \mathbf{R} \to X$ on a compact metric space X "*decomposes*" into a chain recurrent flow and a gradient-like flow. More precisely, he proved the existence of a continuous function $L\colon X \to \mathbf{R}$ which (i) decreases along any orbit of the flow in the complement $X - R$ of the chain recurrent set $R \subset X$ of Φ and (ii) is constant on the connected components of R. Such a function L is called a *Lyapunov function* for Φ. This existence result plays a fundamental role in Conley's program of understanding general flows as collections of isolated invariant sets linked by heteroclinic orbits.

A more general notion of a *Lyapunov* 1*-form* was introduced in paper [5]. Lyapunov 1-forms, as compared to Lyapunov functions, allow one to go one step further and to analyze the flow within the chain-recurrent set R as well. Lyapunov 1-forms provide an important tool in applying methods of homotopy theory to dynamical systems. In the recent papers [4], [5] a generalization of the Lusternik–Schnirelman theory was constructed which applies to flows admitting Lyapunov 1-forms.

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The problem of existence of Lyapunov 1-forms was addressed in our recent preprint [6], where we worked in the category of compact metric spaces, continuous flows and continuous closed 1-forms. In the present paper we study the smooth version of the problem: we construct smooth Lyapunov 1-forms for smooth flows on smooth manifolds. We use Schwartzman's asymptotic cycles to formulate a necessary condition for the existence of Lyapunov 1-forms in a given cohomology class. We also show that under an additional assumption this condition is equivalent to the homological condition introduced in our previous paper [6].

2. Definition

Let V be a smooth vector field on a smooth manifold M. Assume that V generates a continuous flow $\Phi\colon M \times \mathbf{R} \to M$ and $Y \subset M$ is a closed, flow-invariant subset.

DEFINITION 1. A smooth closed 1-form ω on M is called a *Lyapunov 1-form* for the pair (Φ, Y) if it has the following properties:

($\Lambda 1$) The function $\iota_V(\omega) = \omega(V)$ is negative on $M - Y$;

($\Lambda 2$) There exists a smooth function $f\colon U \to \mathbf{R}$ defined on an open neighborhood U of Y such that

$$\omega|_U = df \quad \text{and} \quad df|_Y = 0\,.$$

The above definition is a modification of the notion of a Lyapunov 1-form introduced in section 6 of [5]. The definition of [5] requires that Y consists of finitely many points and the vector field V is locally a gradient of ω with respect to a Riemannian metric.

Definition 1 can also be compared with the definition of a Lyapunov 1-form in the continuous setting which was introduced in [6]. Condition ($\Lambda 1$) above is slightly stronger than condition (L1) of Definition 1 in [6]. Condition ($\Lambda 2$) is similar to condition (L2) of Definition 1 from [6] although they are not equivalent.

There are several natural alternatives for condition ($\Lambda 2$). One of them is:

($\Lambda 2'$) The 1-form ω, viewed as a map $\omega\colon M \to T^*(M)$, vanishes on Y.

It is clear that ($\Lambda 2$) implies ($\Lambda 2'$). We can show that the converse is true under some additional assumptions:

LEMMA 1. *If the de Rham cohomology class* ξ *of* ω *is integral,* $\xi = [\omega] \in H^1(M;\mathbf{Z})$, *then the conditions* ($\Lambda 2'$) *and* ($\Lambda 2$) *are equivalent.*

Proof. Clearly we only need to show that ($\Lambda 2'$) implies ($\Lambda 2$). Since ξ is integral there exists a smooth map $\phi\colon M \to S^1$ such that $\omega = \phi^*(d\theta)$, where $d\theta$ is the standard angular 1-form on the circle S^1. Let $\alpha \in S^1$ be a regular value of ϕ. Assuming that ($\Lambda 2'$) holds it then follows that $U = M - \phi^{-1}(\alpha)$ is an open neighborhood of Y. Clearly $\omega|_U = df$ where $f\colon U \to \mathbf{R}$ is a smooth function which is related to ϕ by $\phi(x) = \exp(if(x))$ for any $x \in U$. Hence ($\Lambda 2$) holds.

LEMMA 2. *The conditions* ($\Lambda 2'$) *and* ($\Lambda 2$) *are equivalent if* Y *is an Euclidean Neighborhood Retract (ENR).*

Proof. Again, we only have to establish ($\Lambda 2'$) $\Rightarrow$ ($\Lambda 2$). Since Y is an ENR it admits an open neighbourhood $U \subset M$ such that the inclusion $i_U\colon U \to M$ is homotopic to $i_Y \circ r$, where $i_Y\colon Y \to M$ is the inclusion and $r\colon U \to Y$ is a retraction (see [3], chapter 4, §8, Corollary 8.7). Pick a base point x_j in every path-connected component U_j of U and define a smooth function $f_j\colon U_j \to \mathbf{R}$ by

$$f_j(x) = \int_{x_j}^{x} \omega\,, \qquad x \in U_j\,.$$

The latter integral is independent of the choice of the integration path in U_j connecting x_j with x. This claim is equivalent to the vanishing of the integral $\int_\gamma \omega$ for any closed loop γ lying in U. To show this we apply the retraction to see that γ is homotopic in M to the loop $\gamma_1 = r \circ \gamma$, which lies in Y; thus we obtain $\int_\gamma \omega = \int_{\gamma_1} \omega = 0$ because of ($\Lambda 2'$). It is clear that the functions f_j together determine a smooth function $f\colon U \to \mathbf{R}$ with $df = \omega|_U$.

A class of interesting examples can be obtained as follows. Let ω be a smooth closed 1-form on a closed Riemannian manifold M. Consider the negative gradient vector field V of ω, i.e. $\langle V, X\rangle = -\omega(X)$ for any vector field X on M where $\langle \cdot,\cdot \rangle$ denotes the Riemannian metric. Denote by Φ the flow induced by the vector field V and by Y the set of zeros of ω. Then clearly conditions ($\Lambda 1$) and ($\Lambda 2'$) are satisfied. If either the cohomology class of ω is integral or Y is an ENR then (by the two Lemmas above) ω is a Lyapunov 1-form for the pair (Φ, Y).

Our main goal in this paper is to find topological conditions which guarantee that for a given vector field V on M there exists a Lyapunov 1-form ω lying in a prescribed cohomology class $\xi \in H^1(M;\mathbf{R})$.

3. Asymptotic cycles of Schwartzman

Let M be a closed smooth manifold and let V be a smooth vector field. Let $\Phi\colon M \times \mathbf{R} \to M$ be the flow generated by V.

Consider a Borel measure μ on M which is invariant under Φ. According to S. Schwartzman [16], these data determine a real homology class

$$\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M;\mathbf{R})$$

called *the asymptotic cycle of the flow* Φ *corresponding to the measure* μ. The class $\mathcal{A}_\mu$ is defined as follows. For a de Rham cohomology class $\xi \in H^1(M;\mathbf{R})$ the evaluation $\langle \xi, \mathcal{A}_\mu \rangle \in \mathbf{R}$ is given by the integral

$$\langle \xi, \mathcal{A}_\mu \rangle = \int_M \iota_V(\omega)\, d\mu\,, \tag{3.1}$$

where ω is a closed 1-form in the class ξ. Note that $\langle \xi, \mathcal{A}_\mu \rangle$ is well-defined, i.e. it depends only on the cohomology class ξ of ω, see [16], p. 277. Indeed, replacing ω by $\omega' = \omega + df$, where $f\colon M \to \mathbf{R}$ is a smooth function, the integral in (3.1) gets changed by the quantity

$$\int_M V(f)\, d\mu = \lim_{s \to 0} \frac{1}{s} \int_M \{f(x\cdot s) - f(x)\}\, d\mu(x)\,. \tag{3.2}$$

Here $V(f)$ denotes the derivative of f in the direction of the vector field V and $x \cdot s$ stands for the flow $\Phi(x,s)$ of the vector field V. Since the measure μ is flow invariant, the integral on the RHS of (3.2) vanishes for any f. It is clear that the RHS of (3.1) is a linear function of $\xi \in H^1(M;\mathbf{R})$. Hence there exists a unique real homology class $\mathcal{A}_\mu \in H_1(M;\mathbf{R})$ which satisfies (3.1) for all $\xi \in H^1(M;\mathbf{R})$.

4. Necessary conditions

We consider the flow Φ as being fixed and we vary the invariant measure μ. As the class $\mathcal{A}_\mu \in H_1(M;\mathbf{R})$ depends linearly on μ, the set of asymptotic cycles $\mathcal{A}_\mu$ corresponding to all Φ-invariant positive measures μ forms a convex cone in the vector space $H_1(M;\mathbf{R})$.

PROPOSITION 1. *Assume that there exists a Lyapunov* 1*-form for* (Φ, Y) *lying in a cohomology class* $\xi \in H^1(M;\mathbf{R})$. *Then*

$$\langle \xi, \mathcal{A}_\mu \rangle \leq 0 \tag{4.1}$$

for any Φ*-invariant positive Borel measure* μ *on* M*; equality in (4.1) takes place if and only if the complement of* Y *has measure zero. Further, the restriction of* ξ *to* Y*, viewed as a Čech cohomology class*

$$\xi|_Y \in \check{H}^1(Y;\mathbf{R})$$

vanishes, $\xi|_Y = 0$.

Proof. Let ω be a Lyapunov 1-form for (Φ, Y) lying in the class ξ. According to Definition 1, the function $\iota_V(\omega)$ is negative on $M - Y$ and vanishes on Y. We obtain that the integral

$$\int_M \iota_V(\omega)\, d\mu = \langle \xi, \mathcal{A}_\mu \rangle$$

is nonpositive.

Assuming $\mu(M-Y) > 0$, we find a compact $K \subset M - Y$ with $\mu(K) > 0$; this follows from the Theorem of Riesz – see e.g. [12], Theorem 2.3(iv), p. 256. There is a constant $\epsilon > 0$ such that $\iota_V(\omega)|_K \leq -\epsilon$. Therefore, one has

$$\int_M \iota_V(\omega)\, d\mu \leq -\epsilon\mu(K) < 0\,.$$

Hence, the value $\langle \xi, \mathcal{A}_\mu \rangle$ is strictly negative if the measure μ is not supported in Y.

To prove the second statement we observe (see [19]) that the Čech cohomology $\check{H}^1(Y;\mathbf{R})$ equals the direct limit of the singular cohomology

$$\check{H}^1(Y;\mathbf{R}) = \varinjlim_{W \supset Y} H^1(W;\mathbf{R}),$$

where W runs over open neighborhoods of Y. It is clear in view of condition $(\Lambda 2)$ that $\xi|_U = 0 \in H^1(U;\mathbf{R})$ (by the de Rham theorem). Hence the result follows.

5. CHAIN-RECURRENT SET R_ξ

Given a flow Φ, our aim is to construct a Lyapunov 1-form ω for a pair (Φ, Y) lying in a given cohomology class $\xi \in H^1(M;\mathbf{R})$. A natural candidate for Y is the subset $R_\xi = R_\xi(\Phi)$ of the chain-recurrent set $R = R(\Phi)$ which was defined in [6]. For convenience of the reader we briefly recall the definition.

Fix a Riemannian metric on M and denote by d the corresponding distance function. Given any $\delta > 0$, $T > 1$, a *(δ, T)-chain from $x \in M$ to $y \in M$* is a finite sequence $x_0 = x, x_1, \ldots, x_N = y$ of points in M and numbers $t_1, \ldots, t_N \in \mathbf{R}$ such that $t_i \geq T$ and $d(x_{i-1} \cdot t_i, x_i) < \delta$ for all $1 \leq i \leq N$. Here we use the notation $\Phi(x,t) = x \cdot t$. The *chain recurrent set* $R = R(\Phi)$ of the flow Φ is defined as the set of all points $x \in M$ such that for any $\delta > 0$ and $T > 1$ there exists a (δ, T)-chain starting and ending at x. The chain recurrent set is closed and invariant under the flow.

Given a cohomology class $\xi \in H^1(M; \mathbf{R})$ there is a natural covering space $p_\xi \colon \widetilde{M}_\xi \to M$ associated with ξ. A closed loop $\gamma \colon [0,1] \to M$ lifts to a closed loop in $\widetilde{M}_\xi$ if and only if the value of the cohomology class ξ on the homology class $[\gamma] \in H_1(M; \mathbf{Z})$ vanishes, $\langle \xi, [\gamma] \rangle = 0$. See [19].

The flow Φ lifts uniquely to a flow $\widetilde{\Phi}$ on the covering $\widetilde{M}_\xi$. Consider the chain recurrent set $R(\widetilde{\Phi}) \subset \widetilde{M}_\xi$ of the lifted flow and denote by $R_\xi = p_\xi(R(\widetilde{\Phi})) \subset M$ its projection onto M. The set R_ξ is referred to as *the chain recurrent set associated to the cohomology class* ξ. It is clear that R_ξ is a closed and Φ-invariant subset of R. We denote by C_ξ the complement of R_ξ in R,

$$C_\xi = R - R_\xi\,.$$

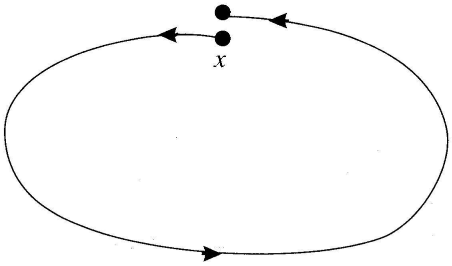
Let us mention the following example illustrating the definition of R_ξ. Consider a smooth flow on a closed manifold M whose chain recurrent set R consists of finitely many rest points and periodic orbits. Given a cohomology class $\xi \in H^1(M; \mathbf{R})$, the chain recurrent set R_ξ is the union of all the rest points and of those periodic orbits whose homology classes $z \in H_1(M; \mathbf{Z})$ satisfy $\langle \xi, z \rangle = 0$.

In general, if the homology class $z \in H_1(M; \mathbf{Z})$ of a periodic orbit satisfies $\langle \xi, z \rangle = 0$ then the orbit belongs to R_ξ. However, it may happen that the points of a periodic orbit belong to R_ξ although $\langle \xi, z \rangle \neq 0$; such an example is described in [6], example after Definition 5.

A different definition of R_ξ which does not use the covering space $\widetilde{M}_\xi$ can be found in [6].

To state our main result we also need the following notion.

A *(δ, T)-cycle* of the flow Φ is defined as a pair (x, t), where $x \in M$ and $t > T$ such that $d(x, x \cdot t) < \delta$. If δ is small enough then any (δ, T)-cycle determines in a canonical way a unique homology class $z \in H_1(M; \mathbf{Z})$ which



is represented by the flow trajectory from x to $x \cdot t$ followed by a "*short*" arc connecting $x \cdot t$ with x. See [6].

6. THEOREM

THEOREM 1. *Let V be a smooth vector field on a smooth closed manifold M. Denote by $\Phi: M \times \mathbf{R} \to M$ the flow generated by V. Let $\xi \in H^1(M;\mathbf{R})$ be a cohomology class such that the restriction $\xi|_{R_\xi}$, viewed as a Čech cohomology class $\xi|_{R_\xi} \in \check{H}^1(R_\xi;\mathbf{R})$, vanishes. Then the following properties of ξ are equivalent:*

(I) *There exists a smooth Lyapunov 1-form for (Φ, R_ξ) in the cohomology class ξ and the subset C_ξ is closed.*

(II) *For any Riemannian metric on M there exist $\delta > 0$ and $T > 1$ such that the homology class $z \in H_1(M;\mathbf{Z})$ associated with an arbitrary (δ, T)-cycle (x,t) of the flow, with $x \in C_\xi$, satisfies $\langle \xi, z\rangle \leq -1$.*

(III) *The subset C_ξ is closed and there exists a constant $\eta > 0$ such that for any Φ-invariant positive Borel measure μ on M the asymptotic cycle $\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M;\mathbf{R})$ satisfies*

$$\langle \xi, \mathcal{A}_\mu \rangle \leq -\eta \cdot \mu(C_\xi)\,. \tag{6.1}$$

(IV) *The subset C_ξ is closed and for any Φ-invariant positive Borel measure μ on X with $\mu(C_\xi) > 0$, the asymptotic cycle $\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M;\mathbf{R})$ satisfies*

$$\langle \xi, \mathcal{A}_\mu \rangle < 0\,. \tag{6.2}$$

The main point of this result is that it gives sufficient homological conditions for the existence of a Lyapunov 1-form in the cohomology class ξ.

Condition (6.1) can be reformulated using the notion of a quasi-regular point. Recall that $x \in X$ is a *quasi-regular* point of the flow $\Phi\colon X \times \mathbf{R} \to X$ if for any continuous function $f\colon X \to \mathbf{R}$ the limit

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(x \cdot s)\, ds \tag{6.3}$$

exists. It follows from the ergodic theorem that the subset $Q \subset X$ of all quasi-regular points has full measure with respect to any Φ-invariant positive Borel measure on X, see [11], p. 106. From the Riesz representation theorem, see e.g. [15], p. 256, one deduces that for any quasi-regular point $x \in Q$ there exists a unique positive flow-invariant Borel measure μ_x with $\mu_x(X) = 1$ satisfying

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(x \cdot s)\, ds = \int_X f\, d\mu_x \tag{6.4}$$

for any continuous function f. We use below the well-known fact that any positive, Φ-invariant Borel measure μ with $\mu(X) = 1$ belongs to the weak* closure of the convex hull of the set of measures μ_x, $x \in Q$, see [11], p. 108.

If the subset $C_\xi \subset X$ is closed, and hence compact, one can apply the above mentioned facts to the restriction of the flow to C_ξ. Let ω be an arbitrary smooth closed 1-form lying in the cohomology class ξ. For any quasi-regular point $x \in C_\xi$ of the flow $\Phi|_{C_\xi}$ one has

$$\lim_{t\to\infty} \frac{1}{t} \int_x^{x\cdot t} \omega = \lim_{t\to\infty} \frac{1}{t} \int_0^t \iota_V(\omega)(x \cdot s)\, ds = \int_M \iota_V(\omega)\, d\mu_x = \langle \xi, \mathcal{A}_{\mu_x} \rangle\,. \tag{6.5}$$

We therefore conclude that condition (III) is equivalent to:

(III$'$) *The subset C_ξ is closed and there exists a constant $\eta > 0$ such that for any quasi-regular point $x \in C_\xi$,*

$$\lim_{t\to\infty} \frac{1}{t} \int_x^{x\cdot t} \omega \le -\eta\,, \tag{6.6}$$

where ω is an arbitrary closed 1-form in the class ξ.

The value of the limit (6.6) is independent of the choice of a closed 1-form ω; the only requirement is that ω lies in the cohomology class ξ.

In the special case $\xi = 0$ the set C_ξ is empty and $R = R_\xi$. The above statement then reduces to the following well-known theorem of C. Conley – see [1] and [18], Theorem 3.14:

PROPOSITION 2 (C. Conley). *Let V be a smooth vector field on a smooth closed manifold M. Denote by $\Phi: M \times \mathbf{R} \to M$ the flow generated by V and by R the chain recurrent set of Φ. Then there exists a smooth Lyapunov function $L: M \to \mathbf{R}$ for (Φ, R). This means that $V(L) < 0$ on $M - R$ and $dL = 0$ pointwise on R.*

Proposition 2 is used in the proof of Theorem 1. As we could not find a proof of this statement in the literature we present one in the appendix.

Next we state a simple corollary of Theorem 1.

COROLLARY 1. *Let $\Phi: M \times \mathbf{R} \to M$ be a smooth flow on a closed manifold M. Any de Rham cohomology class $\xi \in H^1(M; \mathbf{R})$ satisfying*

$$\xi|_R = 0 \in \check{H}^1(M; \mathbf{R}),$$

where $R = R(\Phi)$ denotes the chain recurrent set of the flow, contains a Lyapunov 1-form ω for (Φ, R).

We emphasize that vanishing $\xi|_R = 0$ is supposed to happen in the Čech cohomology. Corollary 1 follows directly from Theorem 1 since under the assumption $\xi|_R = 0$ the set R_ξ coincides with R and so the set C_ξ is empty. Corollary 1 admits also a simple proof independent of Theorem 1 based on Conley's Theorem (Proposition 2 above).

7. EXAMPLES

Here we describe a class of examples of flows for which there exists a cohomology class ξ satisfying all the conditions of Theorem 1.

Let M be a closed smooth manifold with a smooth vector field v. Let $\Psi: M \times \mathbf{R} \to M$ be the flow of v. Assume that the chain recurrent set $R(\Psi)$ is a union of two disjoint closed sets, $R(\Psi) = R_1 \cup R_2$ and $R_1 \cap R_2 = \varnothing$. With these data we will construct a flow Φ on

$$X = M \times S^1$$

such that $R_\xi(\Phi) = R_1 \times S^0$, $C_\xi = R_2 \times S^1$. Here $\xi \in H^1(X; \mathbf{Z})$ denotes the de Rham cohomology class of the 1-form $-d\theta$ where $\theta \in [0, 2\pi]$ denotes the angle coordinate on the circle S^1. $S^0 \subset S^1$ is a two-point set.

We will need two vector fields w_1 and w_2 on S^1, $w_1 = cos(\theta) \cdot \frac{\partial}{\partial\theta}$ and $w_2 = \frac{\partial}{\partial\theta}$. The field w_1 has two zeros $\{p_1, p_2\} = S^0 \subset S^1$ corresponding to the angles $\theta = \pi/2$ and $\theta = 3\pi/2$.

Let $f_i\colon M \to [0,1]$, where $i = 1,2$, be two smooth functions having disjoint supports and satisfying $f_1|_{R_1} = 1$, $f_2|_{R_2} = 1$.

Consider the flow $\Phi\colon X \times \mathbf{R} \to X$ determined by the vector field

$$V = v + f_1 w_1 + f_2 w_2 .$$

Any trajectory of V has the form $(\gamma(t), \theta(t))$, where $\dot\gamma(t) = v(\gamma(t))$, i.e. $\gamma(t)$ is a trajectory of v. It follows that the chain recurrent set of V is contained in $R(\Psi) \times S^1$. Over R_1 we have the vertical vector field w_1 along the circle which has two points $S^0 \subset S^1$ as its chain recurrent set. Over R_2 we have the vertical vector field w_2 which has all of S^1 as the chain recurrent set. We see that $R_1 \times S^0 = R_\xi(\Phi)$, $R_2 \times S^1 = C_\xi$. Hence

$$\xi|_{R_\xi} = 0$$

and C_ξ is closed. One easily checks that condition (III) of Theorem 1 (and hence the other conditions as well) is satisfied.

Further examples can be found in section 7 of our paper [6].

8. Proof of Theorem 1

The implication (I) $\Rightarrow$ (II) follows from the proof of Proposition 4 in [6].

(II)$\Rightarrow$(III). By [6], Theorem 2, the set C_ξ is closed. Now we want to show that the inequality (6.2) is satisfied for any positive Φ-invariant Borel measure μ on X with $\mu(C_\xi) > 0$. Fix a closed 1-form ω in the cohomology class ξ. By Lemma 6 from [6], there exist constants $\alpha > 0$ and $\beta > 0$ such that for any $x \in C_\xi$ and $t > 0$, one has

$$\int_x^{x\cdot t} \omega \leq -\alpha t + \beta .$$

Set $t_0 = 2\beta/\alpha$. Then for any $x \in C_\xi$ and $t \geq t_0$ we have

$$\frac{1}{t}\int_x^{x\cdot t} \omega \leq -\frac{\alpha}{2} . \tag{8.1}$$

With any quasi-regular point $x \in C_\xi$ one associates in a canonical way a positive Φ-invariant Borel measure μ_x on C_ξ, see above. It has the property that

(8.2) $$\lim_{t\to\infty}\frac{1}{t}\int_x^{x\cdot t}\omega=\int_M \iota_V(\omega)\,d\mu_x\,.$$

From (8.1) and (8.2) one obtains

(8.3) $$\langle\xi,\mathcal{A}_{\mu_x}\rangle\le-\frac{\alpha}{2}<0$$

for any quasi-regular point $x\in C_\xi$. According to [11], p. 108, any positive Φ-invariant Borel measure μ with $\mu(M)=\mu(C_\xi)=1$ belongs to the weak* closure of the convex hull of the set of measures $\{\mu_x;\ x\in C_\xi \text{ is quasi-regular}\}$; hence

(8.4) $$\langle\xi,\mathcal{A}_{\mu}\rangle\le-\frac{\alpha}{2}<0\,.$$

It is well known that every (finite) positive Φ-invariant Borel measure is supported on $R=R_\xi\cup C_\xi$, see e.g. [13], Proposition 4.1.18, p. 141. As R_ξ and C_ξ are closed and flow-invariant we may write $\mu=\mu_1+\mu_2$ where μ_1,μ_2 are Φ-invariant and μ_1 is supported on R_ξ, while μ_2 is supported on C_ξ. It follows from (8.4) that $\langle\xi,\mathcal{A}_{\mu_2}\rangle\le-\frac{\alpha}{2}\cdot\mu_2(C_\xi)$. Further, we claim that $\langle\xi,\mathcal{A}_{\mu_1}\rangle=0$ for the following reason. Since $\xi|_{R_\xi}=0$ (as a Čech cohomology class), for any smooth closed 1-form ω on M representing ξ there exists a smooth function f defined on an open neighborhood of R_ξ such that $\omega=df$ near R_ξ. Then we obtain

(8.5) $$\langle\xi,\mathcal{A}_{\mu_1}\rangle=\int_M\iota_V(\omega)\,d\mu_1=\int_{R_\xi}\iota_V(\omega)\,d\mu_1=\int_{R_\xi}V(f)\,d\mu_1=0\,.$$

The last equality holds since the measure μ_1 is Φ-invariant (see e.g. [16], Theorem on page 277). Finally, as $\mathcal{A}_\mu=\mathcal{A}_{\mu_1}+\mathcal{A}_{\mu_2}$ we see that $\langle\xi,\mathcal{A}_\mu\rangle\le-\eta\cdot\mu(C_\xi)$ with $\eta=\alpha/2$, which completes the proof of (II)⇒(III).

The implication (III)⇒(IV) is obvious.

We are left to show the implication (IV)⇒(I). Our argument uses the technique of Schwartzman [16]. It is to show that under the conditions (IV) there exists a smooth Lyapunov 1-form for (Φ,R_ξ) in the class ξ. In a first step we prove that there exists a smooth, closed 1-form ω_1 in the class ξ so that $\iota_V(\omega_1)<0$ on C_ξ. To this end, denote by $\mathcal{D}\subset C^0(M)$ the space of functions

$$\mathcal{D}=\{V(f);\ f\colon M\to\mathbf{R}\text{ is smooth}\}$$

and by $\mathcal{C}^-$ the convex cone in $C^0(M)$ consisting of all functions $f\in C^0(M)$ with

$$f(x)<0\quad\text{for all } x\in C_\xi\,.$$

As C_ξ is compact, the cone $\mathcal{C}^-$ is open in the Banach space $C^0(M)$ of continuous functions on M, endowed with the usual supremum norm. Choose an arbitrary smooth, closed 1-form ω in the class ξ. Assume that $\mathcal{C}^- \cap (\iota_V(\omega) + \mathcal{D}) = \varnothing$. It then follows from the Hahn–Banach Theorem (cf. [15], p. 58) that there exists a continuous linear functional $\Lambda\colon C^0(M) \to \mathbf{R}$ such that

$$\Lambda|_{\iota_V(\omega)+\mathcal{D}} \geq 0 \quad \text{and} \quad \Lambda|_{\mathcal{C}^-} < 0 .$$

Since $\iota_V(\omega)+\mathcal{D}$ is an affine subspace and Λ is bounded on it from below, we obtain that Λ restricted to $\mathcal{D}$ vanishes. According to the Riesz representation theorem (cf. [12]), there exists a Borel measure μ on M so that

$$\Lambda(f) = \int_M f\, d\mu$$

for any $f \in C^0(M)$. By Theorem [16], p. 277, the condition $\Lambda|_{\mathcal{D}} = 0$ implies that μ is Φ-invariant. On the other hand, $\Lambda|_{\mathcal{C}^-} < 0$ implies that $\mu|_{C_\xi} > 0$.

Denote by $\chi\colon M \to \mathbf{R}$ the characteristic function of C_ξ and let $\nu = \chi \cdot \mu$. As C_ξ is Φ-invariant ν is a Φ-invariant Borel measure and (unlike, possibly, μ) is positive. Note that $\mu - \nu$ is a Φ-invariant Borel measure supported on R_ξ (again using that any Φ-invariant measure is supported on $R = R_\xi \cup C_\xi$). Thus, it follows from our assumption $\xi|_{R_\xi} = 0$, by the same argument which led to (8.5), that

$$\langle \xi, \mathcal{A}_{\mu-\nu} \rangle = 0 .$$

Since $\mathcal{A}_{\mu-\nu} = \mathcal{A}_\mu - \mathcal{A}_\nu$ we find

$$\langle \xi, \mathcal{A}_\nu \rangle = \langle \xi, \mathcal{A}_\mu \rangle = \int_M \iota_V(\omega)\, d\mu = \Lambda(f) \geq 0$$

where $f = \iota_V(\omega)$, contradicting condition (6.2). This means that the intersection $\mathcal{C}^- \cap (\iota_V(\omega)+\mathcal{D})$ cannot be empty, i.e. there exists a smooth function $g\colon M \to \mathbf{R}$ so that the smooth closed 1-form $\omega_1 = \omega + dg$ is in the class ξ and satisfies

$$\iota_V(\omega_1) < 0 \quad \text{on } C_\xi .$$

This completes the first step of the proof.

To finish the argument, we now adjust ω_1 on the complement of C_ξ so that the resulting form is a Lyapunov 1-form for (Φ, R_ξ). As $\iota_V(\omega_1) < 0$ on C_ξ and C_ξ is compact, there is some open neighborhood W_1 of C_ξ such that $W_1 \cap R_\xi = \varnothing$ and $\iota_V(\omega_1) < 0$ on W_1. Since $\xi|_{R_\xi} = 0$, there exists an open neighborhood W_2 of R_ξ such that $W_1 \cap W_2 = \varnothing$ and a smooth function $g\colon M \to \mathbf{R}$ such that $\omega_{1|W_2} = dg$ and $dg|_{W_1} = 0$. By Proposition 2 there exists a smooth Lyapunov function $L\colon M \to \mathbf{R}$ for (Φ, R). Now consider

(8.6) $$\omega_2 = \omega_1 - dg + \lambda dL\,,$$

where $\lambda > 0$ remains to be chosen. Clearly, the form ω_2 is smooth and closed and represents the class ξ. For any $\lambda > 0$ it satisfies $\omega_2|_{W_2} = d(\lambda L)$, because $\omega_1 - dg$ vanishes on this set. In particular, ω_2 has property (Λ2) of a Lyapunov 1-form for the pair (Φ, R_ξ). Note also that for all positive λ we have $\iota_V(\omega_2) < 0$ on W_1 (by the construction of W_1) and on $W_2 - R_\xi$ because $\omega_1 - dg$ vanishes there, whereas $V(L) < 0$. As the complement of $W_1 \cup W_2$ is compact and disjoint from R,

$$1 < \lambda_0 := 1 + \sup_{x \notin W_1 \cup W_2} \frac{|\iota_V(\omega_1 - dg)|}{|V(L)|} < \infty\,,$$

and $\iota_V(\omega_2) < 0$ on $M - R_\xi$ for all $\lambda \geq \lambda_0$, showing that for such choices of λ the form ω_2 also has property (Λ1) of a Lyapunov 1-form for (Φ, R_ξ). This completes the proof of the implication (IV)$\Rightarrow$(I) and hence the proof of Theorem 1. □

APPENDIX: PROOF OF PROPOSITION 2

Recall from [1, II 6.2.A] the alternative characterization of the chain recurrent set R as

$$R = \bigcap \{A \cup A^* \mid (A, A^*) \text{ is an attractor-repeller pair}\}.$$

Here a closed, flow-invariant subset $A \subset M$ is called an attractor if it admits a neighborhood U such that A is the maximal flow-invariant subset in the closure of $U \cdot [0, \infty)$. The dual repeller A^* is the set of all points $x \in M$ whose forward limit set is disjoint from A (cf. [1, II 5.1]). Equivalently, (A, A^*) is an attractor-repeller pair if and only if both A and A^* are closed flow invariant subsets of M and the forward (resp. backward) limit set of every point $x \notin A \cup A^*$ is contained in A (resp. A^*) – see [14, Prop. 1.4].

As M is a closed manifold and hence separable, the number of distinct attractor-repeller pairs is at most countable (cf. [1, II 6.4.A]). Let $\{(A_n, A_n^*)\}_{n \geq 1}$ be some enumeration. For each $n \geq 1$, the construction of Robbin and Salamon (Prop. 1.4. of [14] and the remark following it) yields a smooth function $f_n \colon M \to [0,1]$ with $f_n^{-1}(0) = A_n$, $f_n^{-1}(1) = A_n^*$ and $df_n(V) < 0$ on the complement of $A_n \cup A_n^*$. Let c_n be positive constants such that in a fixed finite atlas of charts all partial derivatives of f_n of order $\leq n$ are bounded pointwise in absolute value by c_n. Then

$$L(x) := \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n c_n}$$

is a smooth function having the required properties. In particular, as for any $n \geq 1$ the differential df_n vanishes on $A_n \cup A_n^*$, the differential of L vanishes on R. □

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