In which dimensions does a division algebra over a given ground field exist?

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IN WHICH DIMENSIONS DOES A DIVISION ALGEBRA OVER A GIVEN GROUND FIELD EXIST?

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ABSTRACT. For any ground field k we denote by $\mathcal{N}(k)$ the set of all natural numbers n such that a division algebra (not assumed to be associative) of dimension n over k exists. We prove that $\mathcal{N}(k) = \{1\}$ if k is algebraically closed, $\mathcal{N}(k) = \{1,2,4,8\}$ if k is real closed, and $\mathcal{N}(k)$ is unbounded if k is neither algebraically closed nor real closed.

1. INTRODUCTION

Throughout this article, k denotes a field.

The class of all fields can be partitioned into the three subclasses of all algebraically closed fields, all real closed fields, and all remaining fields respectively [1]). Is this partition "natural"?

A first affirmative answer is obtained if one considers the degree of the algebraic closure $\bar{k}$ of an arbitrary field k, recalling that

$$[\bar{k}:k] = \begin{cases} 1 & \text{if } k \text{ is algebraically closed} \\ 2 & \text{if } k \text{ is real closed} \\ \infty & \text{if } k \text{ is non-closed}. \end{cases}$$

A second affirmative answer is obtained if one considers the set $\mathcal{M}(k)$ of all degrees of irreducible polynomials in $k[X]$, recalling that

$$\mathcal{M}(k) = \begin{cases} \{1\} & \text{if } k \text{ is algebraically closed} \\ \{1,2\} & \text{if } k \text{ is real closed} \\ \text{unbounded} & \text{if } k \text{ is non-closed}. \end{cases}$$

[1]) For the sake of brevity we shall call a field *non-closed* if it is neither algebraically closed nor real closed.

In both cases the first identity holds by definition of an algebraically closed field, the second identity expresses a basic result from the theory of real closed fields, and the third identity is part of the Artin-Schreier theorem (cf. Proposition 2.4).

A third and even more distinct affirmative answer to our initial question is obtained if one considers the set $\mathcal{N}(k)$ of all dimensions of finite dimensional division algebras over k. It is formulated in Theorem 1.1 below.

Recall that a k-algebra is a vector space A over k which is endowed with a k-bilinear multiplication $A \times A \to A$, $(x,y) \mapsto xy$. Every $x \in A$ determines linear endomorphisms $L_x \colon A \to A$, $L_x(y) = xy$ and $R_x \colon A \to A$, $R_x(y) = yx$. By a *division algebra* over k we mean a non-zero k-algebra A such that L_x and R_x are bijective for all $x \in A \setminus \{0\}$. Note that a finite dimensional non-zero k-algebra A is a division algebra if and only if it has no zero divisors, i.e. $xy = 0$ implies $x = 0$ or $y = 0$ for all $x, y \in A$. Because infinite dimensional division algebras do not occur in the present article, finite dimensional division algebras will henceforth briefly be called division algebras.

THEOREM 1.1. *Let k be a field. Then*

$$\mathcal{N}(k) = \begin{cases} \{1\} & \textit{if } k \textit{ is algebraically closed} \\ \{1,2,4,8\} & \textit{if } k \textit{ is real closed} \\ \textit{unbounded} & \textit{if } k \textit{ is non-closed}. \end{cases}$$

This result, although easily derived from established mathematical theories, has hitherto seemingly escaped observation. Through the present note we mean to bring it to the attention of a broader mathematical public.

2. PROOF OF THEOREM 1.1

Theorem 1.1 turns out to be an easy consequence of the subsequent Propositions 2.1-2.4 whose background is partly algebraic, partly model theoretic, and partly topological. Three of them (Propositions 2.2-2.4) are to be found in wide-spread mathematical literature and must be considered as classical.

In contrast, Proposition 2.1 hardly seems to be known at all. In view of its fundamental nature and elementary proof, this is particularly surprising. It was first pointed out by P. Gabriel, on the occasion of a talk given by the second author at the University of Zürich in 1994. Gabriel's original argument, involving Kronecker's normal forms for pairs of linear maps between two finite dimensional vector spaces, is to be found in the proof of [12, Proposition 1.1]. Here we present a simplified version of Gabriel's argument which does without normal forms, recently observed by the first author.

PROPOSITION 2.1 (Gabriel 1994). *If k is an algebraically closed field and A is a k-algebra with $1 < \dim A < \infty$, then A has zero divisors.*

Proof. Because $\dim A > 1$, there exist non-proportional vectors $v, w \in A$. If L_v is not bijective, then $\dim A < \infty$ implies that L_v is not injective, and hence $vy = 0$ for some $y \in A \setminus \{0\}$. If L_v is bijective, then the linear endomorphism $L_v^{-1}L_w\colon A \to A$ is well-defined. Since k is algebraically closed, $L_v^{-1}L_w$ has an eigenvalue $\lambda \in k$. Every eigenvector y of $L_v^{-1}L_w$ with eigenvalue λ satisfies the identity $(\lambda v - w)y = 0$. □

Recall that two fields are called *elementarily equivalent* in the language $\mathcal{L}_r = \langle +, \cdot, -, 0, 1 \rangle$ of rings if they satisfy the same first order sentences in this language.

PROPOSITION 2.2 (Tarski 1931). *Any two real closed fields are elementarily equivalent.*

Tarski's original sketch of a proof of Proposition 2.2 is published in [26]. Complete proofs are to be found in [27] and [28]. A reprint of [28] is contained in [6]. For alternative proofs, see also [23], Theorem 2.28 and Corollary 3.18.

PROPOSITION 2.3 (Hopf 1940; Bott, Milnor, Kervaire 1958). *Every real division algebra has dimension 1, 2, 4 or 8.*

Hopf proved in [21] that the dimension of a real division algebra is always a power of 2. Bott, Milnor and Kervaire showed in [4] and [24] independently that the dimension of a real division algebra is always less than or equal to 8. For alternative proofs of Proposition 2.3, see also [14] and [20].

PROPOSITION 2.4 (Artin, Schreier 1927). *For every field k which is not algebraically closed, the following statements are equivalent.*

(i) *$[\bar{k} : k]$ is finite.*

(ii) *$\mathcal{M}(k)$ is bounded.*

(iii) *k is real closed.*

The essence of Proposition 2.4 is contained in [1] and [2], where Artin and Schreier originally developed their theory of real closed fields. Our formulation follows Grillet's exposition in [17], Section 8.4.

In this article we consider 1 to be the least natural number. For each $n \in \mathbf{N}$, we set $\underline{n} = \{1, \dots, n\}$. Moreover we denote by $\psi_n(\overline{x}, \overline{y}, \overline{z})$ the first order formula

$$\left(\bigwedge_{h=1}^{n} \left(\sum_{i,j=1}^{n} x_i z_{hij} y_j = 0 \right) \right) \to \left(\left(\bigwedge_{i=1}^{n} x_i = 0 \right) \vee \left(\bigwedge_{j=1}^{n} y_j = 0 \right) \right)$$

in the free variables x_i, y_j and z_{hij}, where $h, i, j \in \underline{n}$. From $\psi_n(\overline{x}, \overline{y}, \overline{z})$ we build the new formula

$$\varphi_n(\overline{z}) = \forall x_1, \dots, x_n, y_1, \dots, y_n \ \psi_n(\overline{x}, \overline{y}, \overline{z})$$

in the free variables z_{hij}. Accordingly, for each $n \in \mathbf{N}$, the formula $\exists \overline{z} \, \varphi_n(\overline{z})$ is a first order sentence in the language of rings. The relevance of this sequence of sentences to our context is illuminated by the subsequent lemma.[2)]

LEMMA 2.5. *For every field k, the following statements hold true.*

(i) $\{1\} \subset \mathcal{M}(k) \subset \mathcal{N}(k)$.

(ii) *For all $n \in \mathbf{N}$, $n \in \mathcal{N}(k)$ if and only if $k \models \exists \overline{z} \, \varphi_n(\overline{z})$.*

Proof. (i) The polynomial $X \in k[X]$ is irreducible of degree 1, and for each irreducible polynomial $p(X) \in k[X]$ of degree m, the field $k(X)/(p(X))$ is a division algebra over k of dimension m.

(ii) Let $n \in \mathbf{N}$. We denote by $(e_1, \dots, e_n)$ the standard basis in k^n, and by $Alg(k^n)$ the set of all algebra structures on k^n, i.e. the set of all k-bilinear mappings $\alpha \colon k^n \times k^n \to k^n$. Moreover, $k^{n \times n \times n}$ denotes the set of all triple sequences $\overline{a} = (a_{hij})$ in k, where $hij \in \underline{n}^3$. A bijection

$$Alg(k^n) \to k^{n \times n \times n}, \quad \alpha \mapsto \overline{a}$$

is given by $\alpha(e_i, e_j) = \sum_{h=1}^{n} a_{hij} e_h$ for all $ij \in \underline{n}^2$. If α and $\overline{a}$ correspond under this bijection, then $\alpha(x, y) = \sum_{h=1}^{n} \left(\sum_{i,j=1}^{n} x_i a_{hij} y_j \right) e_h$ holds for all $x, y \in k^n$. Accordingly we obtain the following chain of equivalences.

$$\begin{aligned} n \in \mathcal{N}(k) &\Leftrightarrow \exists \alpha \in Alg(k^n) \quad \forall x, y \in k^n \ (\alpha(x, y) = 0 \Rightarrow x = 0 \vee y = 0) \\ &\Leftrightarrow k \models \varphi_n(\overline{a}) \text{ for some } \overline{a} \in k^{n \times n \times n} \\ &\Leftrightarrow k \models \exists \overline{z} \, \varphi_n(\overline{z}) . \qquad \square \end{aligned}$$

[2)] If k is a field and φ is a first order sentence in the language of rings, then $k \models \varphi$ expresses that φ is true in k.

Now Theorem 1.1 emerges from Propositions 2.1-2.4, tied up with Lemma 2.5, as a mere corollary.

Proof of Theorem 1.1. If k is algebraically closed and A is a division algebra over k, then $0 < \dim A$ by definition of a division algebra, and $\dim A < 2$ by Proposition 2.1. Hence $\dim A = 1$. Accordingly $\mathcal{N}(k) = \{1\}$.

If k is real closed, then k and $\mathbf{R}$ are elementarily equivalent by Proposition 2.2. With Lemma 2.5.(ii) we conclude that the equivalences

$$n \in \mathcal{N}(k) \Leftrightarrow k \models \exists \bar{z}\, \varphi_n(\bar{z}) \Leftrightarrow \mathbf{R} \models \exists \bar{z}\, \varphi_n(\bar{z}) \Leftrightarrow n \in \mathcal{N}(\mathbf{R})$$

hold for all $n \in \mathbf{N}$. Thus $\mathcal{N}(k) = \mathcal{N}(\mathbf{R})$. Moreover $\mathcal{N}(\mathbf{R}) \subset \{1,2,4,8\}$ by Proposition 2.3, and the classical examples of real division algebras $\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ show that $\{1,2,4,8\} \subset \mathcal{N}(\mathbf{R})$. So $\mathcal{N}(k) = \mathcal{N}(\mathbf{R}) = \{1,2,4,8\}$.

If k is non-closed, then $\mathcal{M}(k)$ is unbounded by Proposition 2.4. With Lemma 2.5.(i) we conclude that $\mathcal{N}(k)$ is unbounded. □

3. Supplement

The interested reader may wonder whether Theorem 1.1 can be improved towards a precise description of the subsets $\mathcal{M}(k) \subset \mathcal{N}(k)$ of $\mathbf{N}$ for all non-closed fields k. In fact, we feel inclined to conjecture that $\mathcal{M}(k) = \mathbf{N}$ (and hence $\mathcal{N}(k) = \mathbf{N}$) holds whenever k is non-closed. Standard algebraic theories provide means to verify this conjecture for prominent classes of fields such as e.g. finite fields, algebraic number fields, local fields, and rational function fields. For non-closed fields in general, however, we do not know any proof of our conjecture, nor any counterexample to it.

Instead, the following supplement to Lemma 2.5(i), approximating $\mathcal{N}(k)$ from above in terms of $\mathcal{M}(k)$ for any field k, may be worthwhile mentioning. For any subset $M \subset \mathbf{N}$ we define

$$\sum M = \left\{ \sum_{i=1}^{\ell} m_i \;\middle|\; m_i \in M \quad \forall i \in \underline{\ell},\ \ell \in \mathbf{N} \right\}.$$

PROPOSITION 3.1. *For every field k, the inclusions*

$$\mathcal{M}(k) \subset \mathcal{N}(k) \subset \{1\} \cup \sum \big(\mathcal{M}(k) \setminus \{1\}\big)$$

hold true.

Proof. In view of Lemma 2.5(i) we only have to prove the second inclusion. If $n \in \mathcal{N}(k)$, then there exists an n-dimensional division algebra A over k. If $n = 1$, then the second inclusion holds true. If $n > 1$, choose non-proportional vectors $v, w \in A$. Since A is a division algebra, the linear endomorphism $L_v \colon A \to A$, $L_v(y) = vy$ is bijective, and hence $L_v^{-1}L_w \colon A \to A$ is well-defined. The characteristic polynomial $\chi = \det(X\mathbf{I}_A - L_v^{-1}L_w)$ of $L_v^{-1}L_w$ has degree n and factors into a product $\chi = \prod_{i=1}^{\ell} p_i$ of monic irreducible polynomials $p_i \in k[X]$. If some p_i is linear, say $p_i = X - \lambda$, then λ is an eigenvalue of $L_v^{-1}L_w$, and hence every eigenvector u of $L_v^{-1}L_w$ with eigenvalue λ satisfies

$$L_v^{-1}L_w(u) = \lambda u \;\Rightarrow\; wu = \lambda vu \;\Rightarrow\; (w - \lambda v)u = 0,$$

contradicting the fact that A has no zero divisors. Accordingly, all p_i are non-linear. Hence $n = \deg \chi = \sum_{i=1}^{\ell} \deg p_i \in \sum\big(\mathcal{M}(k) \setminus \{1\}\big)$. □

Combined with the information on $\mathcal{M}(k)$ presented in the introduction, Proposition 3.1 has the following immediate consequences. For algebraically closed fields k we obtain an alternative proof of the first statement $\mathcal{N}(k) = \{1\}$ in Theorem 1.1, and thereby also an alternative proof of Proposition 2.1. For real closed fields k we obtain the weak (but easily gotten) approximation $\mathcal{N}(k) \subset \{1\} \cup 2\mathbf{N}$ to the second statement $\mathcal{N}(k) = \{1, 2, 4, 8\}$ in Theorem 1.1. For non-closed fields k we obtain an improvement of the third statement $|\mathcal{N}(k)| = \infty$ in Theorem 1.1 by the estimate $\mathcal{N}(k) \subset \{1\} \cup \sum\big(\mathcal{M}(k) \setminus \{1\}\big)$. Of course, if our conjecture is true, then this improvement is void.

4. Epilogue

As soon as $\mathcal{N}(k)$ is known for any given ground field k, the problem of classifying for each $d \in \mathcal{N}(k)$ all d-dimensional division algebras over k up to isomorphism, arises naturally. This problem is trivial for $d = 1$, difficult for $d = 2$, and very hard for $d > 2$. Let us add some information in support of this judgement. The category of all d-dimensional division algebras over k is denoted by $\mathcal{D}_d(k)$.

$\mathcal{D}_1(k)$ is classified by $\{k\}$, for every field k. Indeed, given any $A \in \mathcal{D}_1(k)$, we may choose $a \in A \setminus \{0\}$. Then $a^2 = \alpha a$ for some $\alpha \in k$. Now $e = \alpha^{-1}a$

is a non-zero idempotent in A. Accordingly, the linear map $A \to k$, $e \mapsto 1$ is an algebra isomorphism. Hence $\mathcal{D}_1(k)$ consists of the isoclass of k only.

The problem of classifying $\mathcal{D}_2(k)$ has been solved for $k = \mathbf{R}$. Classifying subsets of $\mathcal{D}_2(\mathbf{R})$ are to be found in [5], [16], [22] and [13]. Their description involves up to 4 independent continuous real parameters.

In case $d > 2$ and k is any field such that $d \in \mathcal{N}(k)$, no complete solution to the problem of classifying $\mathcal{D}_d(k)$ is known to date. There are however partial solutions in the classical case $k = \mathbf{R}$. Noticeable among them are the classification of all 4-dimensional real *quadratic* division algebras $\mathcal{D}_4^q(\mathbf{R})$, initiated by Osborn [25], continued by Hefendehl-Hebeker [18], [19] and accomplished by Dieterich [10], [11], [12], as well as the classification of all real *flexible* division algebras $\mathcal{D}^f(\mathbf{R})$, initiated by Benkart, Britten and Osborn [3], continued by Cuenca Mira et al. [7], and accomplished by Darpö [8], [9]. The description of a classifying subset of $\mathcal{D}_4^q(\mathbf{R})$ involves up to 9 independent continuous real parameters. While both $\mathcal{D}_2^f(\mathbf{R})$ and $\mathcal{D}_4^f(\mathbf{R})$ are classified by 2-parameter families, the description of a classifying subset of $\mathcal{D}_8^f(\mathbf{R})$ involves up to 15 independent continuous real parameters.

A key idea in our proof of Theorem 1.1 was to translate the statement $\mathcal{N}(\mathbf{R}) = \{1,2,4,8\}$ into a sequence of first order sentences in the language of rings, and to conclude with Proposition 2.2 that this statement remains true when $\mathbf{R}$ is replaced by any real closed field k. Applying the same argument it is straightforward to prove for any real closed field k the following three theorems, using that they are established in case $k = \mathbf{R}$.

THEOREM 4.1. *Every commutative division algebra over a real closed field k has dimension 1 or 2.*

In case $k = \mathbf{R}$, Theorem 4.1 is due to Hopf [21].

THEOREM 4.2. *For every real closed field k there are precisely three isoclasses of associative division algebras over k, one in each of the dimensions 1, 2 and 4.*

In case $k = \mathbf{R}$, Theorem 4.2 is due to Frobenius [15].

THEOREM 4.3. *For every real closed field k there are precisely four isoclasses of alternative division algebras over k, one in each of the dimensions 1, 2, 4 and 8.*

In case $k = \mathbf{R}$, Theorem 4.3 is due to Zorn [29].

The proofs of Proposition 2.3 and Theorem 4.1 in the special case $k = \mathbf{R}$ are strongly based on the topology of $\mathbf{R}^n$, and can therefore not easily be initiated for arbitrary real closed fields. Instead, Tarski's method enables us to use these results without considering their proofs.

The question arises to what extent Tarski's method can be used to deduce results on other classes of algebras over real closed fields, and possibly even over other types of ground fields. To our knowledge, no systematic attempt in this direction has hitherto been made.

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