L'Enseignement Mathématique, t. 51 (2005), p. 331–334

ON PRODUCTS OF DISJOINT BLOCKS OF CONSECUTIVE INTEGERS

by Maciej ULAS

ABSTRACT. In this note we disprove the conjecture of Erdős and Graham which states that for fixed $k \geq 2$ and $l \geq 4$ the product of k disjoint blocks of l consecutive integers is a square only finitely many times. We give two infinite families of solutions for $k = l = 4$.

Let $l \geq 2$ be an integer, and set $f(x) = (x+1) \cdot \ldots \cdot (x+l)$. For a fixed integer $k \geq 1$ we consider the diophantine equation

$$y^2 = \prod_{i=1}^{k} f(x_i)\,, \tag{1}$$

with the condition

$$0 < x_1 < \cdots < x_k\,, \quad x_j + l \leq x_{j+1}\,, \quad j = 1, \ldots, k-1\,. \tag{2}$$

Condition (2) ensures that the blocks $f(x_1), \ldots, f(x_k)$ of consecutive integers are disjoint.

Erdős and Selfridge [1] have proved a celebrated theorem implying that for $k = 1$ and each $l \geq 2$ the only solutions of (1) are $x_1 = -1, \ldots, -l$, $y = 0$. On the other hand, if there is more than one block we have

THEOREM 1. *For $l \leq 3$ and each $k \geq 2$, equation* (1) *has infinitely many solutions satisfying condition* (2).

Proof. For $l = 2$ and $k = 2$ we take

$$x_1 = n-1\,, \quad x_2 = 4n^2 + 4n - 1\,, \quad n \geq 2\,.$$

For $l = 2$ and $k = 3$ we take

$$x_1 = n-1\,, \quad x_2 = 3n\,, \quad x_3 = \frac{9n(n+1)}{2} - 1\,, \quad n \geq 2\,.$$

Since each $k \geq 2$ is of the form $2a+3b$ with nonnegative a, b, we obtain our conclusion for $l=2$.

For $l=3$ and $k=2$, K. R. S. Sastry takes

$$x_1 = n-1\,, \quad x_2 = 2n-1\,,$$

where $n,\ m \in \mathbf{N}$ satisfy the equation

$$(n+2)(2n+1) = m^2\,,$$

which has infinitely many solutions [3].

For $l=3$ and $k=3$ we take, as does Skałba in [4],

$$x_1 = F_{2u-1}-2\,, \quad x_2 = F_{2u+1}-2\,, \quad x_3 = F_{2u}^2-2\,, \quad u \geq 2\,,$$

where F_n is the n-th term of the Fibonacci sequence.

As in the case $l=2$, we see that for each $k \geq 2$ equation (1) has infinitely many solutions satisfying (2). □

In contrast to this result Erdős and Graham conjecture in ([2], p. 67) that if $l \geq 4$, then for each fixed k, equation (1) has only finitely many solutions satisfying (2). This conjecture is also mentioned in [3] as Problem D17.

In connection with this problem M. Skałba proved in [4] that if we allow k to vary, the above conjecture is not true.

We disprove the Erdős and Graham conjecture by establishing the following

THEOREM 2. *For* $k=l=4$, *equation* (1) *has infinitely many solutions satisfying* (2).

Proof. Let n be an integer greater than 1. We put

$$x_1 = 4n-1\,, \quad x_2 = 4n+3\,, \quad x_3 = 4n^2+7n-1\,, \quad x_4 = 8n^2+14n+1\,.$$

Then

$$y^2 = f(x_1)f(x_2)f(x_3)f(x_4)\,,$$

where

$$y = 16n(n+1)(2n+1)(2n+3)(4n+1)(4n+3) \times (4n+5)(4n+7)(4n^2+7n+1)(4n^2+7n+2)\,.$$

We can also take

$$x_1 = 4n\,, \quad x_2 = 4(n+1)\,, \quad x_3 = 4n^2+9n+1\,, \quad x_4 = 8n^2+18n+5\,,$$

and then

$$y = 16(n+1)(n+2)(2n+1)(2n+3)(4n+1)(4n+3) \times (4n+5)(4n+7)(4n^2+9n+3)(4n^2+9n+4)\,. \qquad \square$$

In connection with the above theorem it is natural to ask which is the smallest integer k such that equation (1) with $l = 4$ has infinitely many solutions satisfying (2).

We looked for solutions with $l = 4$ and $k = 2$ or $k = 3$ using a computer.

For $k = 2$ we computed all solutions of (1) which satisfy $x_1+3 < x_2 < 10^5$. In this range we found only one solution, $x_1 = 32$, $x_2 = 1679$.

For $k = 3$ we computed all solutions of (1) which satisfy condition (2) and $x_3 < 5 \cdot 10^3$. The results are shown in the following table.

x_1	x_2	x_3
3	12	23
3	15	167
4	13	47
7	24	62
7	38	285
9	244	1022
10	66	2207
11	30	152
17	30	339
17	47	167

x_1	x_2	x_3
19	95	482
45	158	844
64	74	132
131	321	2207
186	208	421
245	283	494
368	404	898
491	549	1103
920	1063	1841

Since each $k \geq 6$ is of the form $4s$, $4s+2$, $4s+3$ or $4s+5$, we obtain from the above computations and parametric solution in Theorem 2

THEOREM 3. *For* $l = 4$ *and any fixed* $k \geq 6$, *equation* (1) *has infinitely many solutions satisfying* (2).

I firmly believe that in the remaining case $k = 5$ there are infinitely many solutions. In the case $l \geq 5$ the following conjecture seems to be true.

CONJECTURE 4. *For each* $l \geq 5$ *there exists an integer* $n = n(l)$ *such that for each* $k \geq n$ *equation* (1) *has infinitely many solutions satisfying condition* (2).

ACKNOWLEDGEMENTS. I would like to thank Prof. K. Rusek and Prof. A. Schinzel for remarks improving the presentation.

REFERENCES

[1] ERDŐS, P. and J. L. SELFRIDGE. The product of consecutive integers is never a power. *Illinois J. Math. 19* (1975), 292–301.

[2] ERDŐS, P. and R. L. GRAHAM. *Old and New Problems and Results in Combinatorial Number Theory*. Monograph Enseign. Math. *28*, Geneva, 1980.

[3] GUY, R. K. *Unsolved Problems in Number Theory*. 3rd ed. Springer, 2004.

[4] SKAŁBA, M. Products of disjoint blocks of consecutive integers which are powers. *Colloq. Math. 98* (2003), 1–3.

(Reçu le 7 novembre 2005)

Maciej Ulas
Jagiellonian University
Institute of Mathematics
Reymonta 4
PL-30-059 Kraków
Poland
e-mail: Maciej.Ulas@im.uj.edu.pl