

On products of disjoint blocks of consecutive integers

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ON PRODUCTS OF DISJOINT BLOCKS
OF CONSECUTIVE INTEGERS

by Maciej ULAS

ABSTRACT. In this note we disprove the conjecture of Erdős and Graham which states that for fixed $k \geq 2$ and $l \geq 4$ the product of k disjoint blocks of l consecutive integers is a square only finitely many times. We give two infinite families of solutions for $k = l = 4$.

Let $l \geq 2$ be an integer, and set $f(x) = (x + 1) \cdot \dots \cdot (x + l)$. For a fixed integer $k \geq 1$ we consider the diophantine equation

$$(1) \quad y^2 = \prod_{i=1}^k f(x_i),$$

with the condition

$$(2) \quad 0 < x_1 < \dots < x_k, \quad x_j + l \leq x_{j+1}, \quad j = 1, \dots, k - 1.$$

Condition (2) ensures that the blocks $f(x_1), \dots, f(x_k)$ of consecutive integers are disjoint.

Erdős and Selfridge [1] have proved a celebrated theorem implying that for $k = 1$ and each $l \geq 2$ the only solutions of (1) are $x_1 = -1, \dots, -l$, $y = 0$. On the other hand, if there is more than one block we have

THEOREM 1. *For $l \leq 3$ and each $k \geq 2$, equation (1) has infinitely many solutions satisfying condition (2).*

Proof. For $l = 2$ and $k = 2$ we take

$$x_1 = n - 1, \quad x_2 = 4n^2 + 4n - 1, \quad n \geq 2.$$

For $l = 2$ and $k = 3$ we take

$$x_1 = n - 1, \quad x_2 = 3n, \quad x_3 = \frac{9n(n + 1)}{2} - 1, \quad n \geq 2.$$

Since each $k \geq 2$ is of the form $2a + 3b$ with nonnegative a, b , we obtain our conclusion for $l = 2$.

For $l = 3$ and $k = 2$, K. R. S. Sastry takes

$$x_1 = n - 1, \quad x_2 = 2n - 1,$$

where $n, m \in \mathbf{N}$ satisfy the equation

$$(n + 2)(2n + 1) = m^2,$$

which has infinitely many solutions [3].

For $l = 3$ and $k = 3$ we take, as does Skałba in [4],

$$x_1 = F_{2u-1} - 2, \quad x_2 = F_{2u+1} - 2, \quad x_3 = F_{2u}^2 - 2, \quad u \geq 2,$$

where F_n is the n -th term of the Fibonacci sequence.

As in the case $l = 2$, we see that for each $k \geq 2$ equation (1) has infinitely many solutions satisfying (2). \square

In contrast to this result Erdős and Graham conjecture in ([2], p. 67) that if $l \geq 4$, then for each fixed k , equation (1) has only finitely many solutions satisfying (2). This conjecture is also mentioned in [3] as Problem D17.

In connection with this problem M. Skałba proved in [4] that if we allow k to vary, the above conjecture is not true.

We disprove the Erdős and Graham conjecture by establishing the following

THEOREM 2. *For $k = l = 4$, equation (1) has infinitely many solutions satisfying (2).*

Proof. Let n be an integer greater than 1. We put

$$x_1 = 4n - 1, \quad x_2 = 4n + 3, \quad x_3 = 4n^2 + 7n - 1, \quad x_4 = 8n^2 + 14n + 1.$$

Then

$$y^2 = f(x_1)f(x_2)f(x_3)f(x_4),$$

where

$$y = 16n(n + 1)(2n + 1)(2n + 3)(4n + 1)(4n + 3) \\ \times (4n + 5)(4n + 7)(4n^2 + 7n + 1)(4n^2 + 7n + 2).$$

We can also take

$$x_1 = 4n, \quad x_2 = 4(n + 1), \quad x_3 = 4n^2 + 9n + 1, \quad x_4 = 8n^2 + 18n + 5,$$

and then

$$y = 16(n + 1)(n + 2)(2n + 1)(2n + 3)(4n + 1)(4n + 3) \times (4n + 5)(4n + 7)(4n^2 + 9n + 3)(4n^2 + 9n + 4). \quad \square$$

In connection with the above theorem it is natural to ask which is the smallest integer k such that equation (1) with $l = 4$ has infinitely many solutions satisfying (2).

We looked for solutions with $l = 4$ and $k = 2$ or $k = 3$ using a computer.

For $k = 2$ we computed all solutions of (1) which satisfy $x_1 + 3 < x_2 < 10^5$. In this range we found only one solution, $x_1 = 32, x_2 = 1679$.

For $k = 3$ we computed all solutions of (1) which satisfy condition (2) and $x_3 < 5 \cdot 10^3$. The results are shown in the following table.

x_1	x_2	x_3
3	12	23
3	15	167
4	13	47
7	24	62
7	38	285
9	244	1022
10	66	2207
11	30	152
17	30	339
17	47	167

x_1	x_2	x_3
19	95	482
45	158	844
64	74	132
131	321	2207
186	208	421
245	283	494
368	404	898
491	549	1103
920	1063	1841

Since each $k \geq 6$ is of the form $4s, 4s + 2, 4s + 3$ or $4s + 5$, we obtain from the above computations and parametric solution in Theorem 2

THEOREM 3. *For $l = 4$ and any fixed $k \geq 6$, equation (1) has infinitely many solutions satisfying (2).*

I firmly believe that in the remaining case $k = 5$ there are infinitely many solutions. In the case $l \geq 5$ the following conjecture seems to be true.

CONJECTURE 4. *For each $l \geq 5$ there exists an integer $n = n(l)$ such that for each $k \geq n$ equation (1) has infinitely many solutions satisfying condition (2).*

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