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Autor(en): **Budney, Ryan**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **52 (2006)**

Heft 3-4: **L'enseignement mathématique**

PDF erstellt am: **28.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-2238

L'Enseignement Mathématique (2) 52 (2006), 319–359

JSJ-DECOMPOSITIONS OF KNOT AND LINK COMPLEMENTS IN S^3

by Ryan BUDNEY

ABSTRACT. This paper is a survey of some of the most elementary consequences of the JSJ-decomposition and geometrization for knot and link complements in S^3. Formulated in the language of graphs, the result is the construction of a bijective correspondence between the isotopy classes of links in S^3 and a class of vertex-labelled, finite acyclic graphs, called companionship graphs. This construction can be thought of as a uniqueness theorem for Schubert's 'satellite operations'. We identify precisely which graphs are companionship graphs of knots and links respectively. We also describe how a large family of operations on knots and links affects companionship graphs. This family of operations is called 'splicing' and includes, among others, the operations of: cabling, connect-sum, Whitehead doubling and the deletion of a component.

1. INTRODUCTION

Although the JSJ-decomposition is well-known and frequently used to study 3-manifolds, it has been less frequently used to study knot complements in S^3, perhaps because in this setting it overlaps with Schubert's 'satellite' constructions for knots. This paper studies the global nature of the JSJ-decomposition for knot and link complements in S^3. Much of this article is 'survey' in nature, in the sense that many of the primary results here appear elsewhere in the literature, but not in one place. Frequently we offer new proofs of old results, and we attempt to refer to the first-known appearance of theorems.

Schubert [30] was the first to study incompressible tori in knot complements, which he described in the language of 'satellite operations'. Among other results, Schubert showed that satellite operations could be used to recover his connected-sum decomposition of knots [29]. Unlike the case of the connected-sum, Schubert did not give a full uniqueness theorem for general satellite knots. Waldhausen [35] eventually set up a general theory of incompressible surfaces in 3-manifolds, which led to Jaco, Shalen and Johannson's development of the eponymously named JSJ-decomposition [16, 18] where an appropriate uniqueness theorem was proven. The JSJ-decomposition theorem states that every prime 3-manifold M contains a collection of embedded, incompressible tori $T \subset M$ so that if one removes an open tubular neighbourhood of T from M, the resulting manifold $M|T$ is a disjoint union of Seifert-fibred and atoroidal manifolds. Moreover, if one takes a minimal collection of such tori, they are unique up to isotopy. It is the purpose of this paper to work out the explicit consequences of this theorem and the later developments in Geometrization, for knot and link complements in S^3.

Given a link $L \subset S^3$, with complement $C_L = S^3 \setminus U$ (where U is an open tubular neighbourhood of L), a collection of natural questions one might ask is:

1. Which Seifert-fibred manifolds can be realised as components of $C_L|T$ for T the JSJ-decomposition of a knot or link complement C_L?
2. Which non Seifert-fibred manifolds arise in the same way?
3. How are the above manifolds embedded in S^3?
4. How do they all 'fit together' globally, and what combinations are possible?

We partially answer item 3 first. We prove in Proposition 2 that if M is a compact submanifold of S^3 with ∂M a disjoint union of n embedded tori, if we let p and q be the number of solid tori components and non-trivial knot complement components of $S^3 \setminus int(M)$ respectively, where $p+q=n$, then there exists an embedding $f\colon M \to S^3$ so that $f(M)$ is the complement of an open tubular neighbourhood of an n-component link $L \subset S^3$ which contains a q-component unlink as a sublink. This brings Brunnian properties into the picture. We go on to prove in Proposition 3 that there is a canonical choice for f, thus the study of submanifolds of S^3 with torus boundary reduces in a natural way to link theory in S^3.

Section 3, Proposition 4 answers question 1 by computing which Seifert-fibred manifolds embed in S^3. This allows us to determine the links in S^3 which have Seifert-fibred complements in Proposition 5. Proposition 7 gives conditions on when two Seifert-fibred manifolds can be adjacent in the JSJ-decomposition of a link complement. We end Section 3 with a discussion of the geometric structures on Seifert-fibred link complements.

Section 4 is the heart of the paper where we investigate item 4. The components of $C_L \,|\, T$ naturally form the vertex-set of a partially-directed acyclic graph G_L, called the JSJ-graph of L (Definition 5). In Definition 6 we construct the companionship graph $\mathbf{G}_L$ of a link L by labelling the vertices of G_L with natural 'companion links' to L. We show that $\mathbf{G}_L$ is a complete isotopy invariant of L in Proposition 9. From here we investigate the properties of the graphs $\mathbf{G}_L$. The most basic property of $\mathbf{G}_L$ is that it is a 'splicing diagram' (Definition 8). Roughly, this means that $\mathbf{G}_L$ is a finite, acyclic graph, some of whose edges are oriented, where each vertex is labelled by a link, and each edge of the graph corresponds to a matched pair of components of the links decorating the endpoints of the edge.

We complete the study of companionship graphs of knots first, giving a characterisation of their companionship graphs in Theorem 2, showing, among other things, that the companionship graphs are naturally rooted trees. To characterise companionship trees of knots, we use a notion of splicing (Definition 10) that allows us to inductively construct knots with arbitrarily complicated companionship trees. Returning to links, we show that $\mathbf{G}_L$ satisfies a local Brunnian property (Proposition 15). Splicing diagrams that satisfy this property we call 'valid'. We show in Proposition 16 that valid splicing diagrams correspond bijectively to isotopy classes of collections of disjoint embedded tori in link complements. This allows us to identify in Proposition 18 which valid splice diagrams are companionship graphs. Similarly to the case of links, we use splicing to inductively construct arbitrary companionship graphs. In Proposition 19 we determine the companionship graph of the splice of two arbitrary links.

Thurston's Hyperbolisation Theorem answers question 2, telling us that the interiors of non-Seifert-fibred components of $C_L \,|\, T$ are complete hyperbolic manifolds of finite volume. At the end of Section 4 we briefly mention algorithms for finding the JSJ-decomposition and the geometric structures on the components.

Our definitions and conventions regarding knots and links are:

- A *knot* is a compact, connected, boundaryless, oriented, 1-dimensional sub-manifold of S^3. Thus a knot is diffeomorphic to S^1.
- Given a knot K let $+K = K$ and let $-K$ be the *oppositely-oriented knot*, meaning as unoriented manifolds $-K$ and K are the same, thus $-(-K) = K$ but $-K \neq K$.

- Given a finite set A, a *link L indexed by A* is a disjoint collection of knots $\{L_a : a \in A\}$. Given $A' \subset A$, $L_{A'}$ denotes the sublink of L indexed by A'.
- Given two links L and L' with index-set A, an *isotopy* from L to L' is an orientation-preserving diffeomorphism $f\colon S^3 \to S^3$ such that $f(L_a) = L'_a$ for all $a \in A$. This notion agrees with the traditional notion of isotopy because all orientation-preserving diffeomorphisms of S^3 are isotopic to the identity by Cerf [8].
- Given two links L and L' with index sets A and A', we say L and L' are *unoriented-isotopic* if they are isotopic as unoriented submanifolds of S^3. Stated another way, an orientation-preserving diffeomorphism $f\colon S^3 \to S^3$ is called an *unoriented isotopy* from L to L' if there is a bijection $\sigma\colon A \to A'$ and a function $\epsilon_f\colon A \to \{+,-\}$ such $f(L_a) = \epsilon_f(a)L'_{\sigma(a)}$ for all $a \in A$.
- $D^n = \{x \in \mathbf{R}^n : |x| \leq 1\}$ is the compact unit n-disc.
- Given a solid torus $M \simeq S^1 \times D^2$ in S^3, there are two canonical isotopy classes of unoriented curves in ∂M, the meridian and longitude respectively. The *meridian* is the essential curve in ∂M that bounds a disc in M. The *longitude* is the essential curve in ∂M that bounds a 2-sided surface in $S^3 \setminus int(M)$ (a Seifert surface). If M is a closed tubular neighbourhood of a knot K, the longitude of M is parallel to K thus we give it the induced orientation. We give the meridian m the orientation so that $lk(K,m) = +1$.
- For such standard definitions as *connected-sum*, *splittability*, etc, we will follow the notation of Kawauchi [22].
- Following the conventions of Kanenobu [20] and Debrunner [9], given a link L indexed by A we define $\mathcal{U}_L \subset 2^A$ by the rule $\mathcal{U}_L = \{S \subset A : L_S \text{ is not split}\}$. We say that $\mathcal{U}_L$ is the *Brunnian property* of L.
- $\overline{\mathcal{U}}_L = \{S \subset A : L_S \text{ is an } |S|\text{-component unlink}\}$ we will call the *strong Brunnian property* of L.

Our definitions and conventions regarding 3-manifolds are:

- 3-manifolds are taken to be oriented and are allowed boundary.
- For standard definitions of connected-sum, prime, irreducible, Seifert-fibred, incompressible surface, etc, we will use the conventions of Hatcher [14].
- Given a 3-manifold M and a properly-embedded 2-sided surface $S \subset M$, define $M \,|\, S = \{V \subset M : V = \overline{W}$ where W is a path-component of $M \setminus S\}$. We call the elements of $M \,|\, S$ «*the components of $M \,|\, S$*» or «*the components of M split along S*». If S' is a component of S, then S' is a submanifold of at most two components of $M \,|\, S$.

There are several treatments of the JSJ-decompositions of 3-manifolds. There's the original work of Jaco and Shalen [16] and the simultaneous work of Johannson [18]. Some more recent expositions are available as well. There is Hatcher's notes [14], which this article follows, and also the notes of Neumann and Swarup [26]. As Neumann and Swarup [26] point out (Proposition 4.1) all versions of the decomposition are closely related. In the case of 3-manifolds M with $\chi(M) = 0$ such as link complements in S^3, all versions of the JSJ-decomposition are the same. When $\chi(M) \neq 0$ one can get various different incompressible annuli in the decompositions, depending on whose conventions are followed. This difference is important as the original JSJ-decomposition is more closely related to Thurston's geometric decomposition of 3-manifolds.

There are also treatments of the JSJ-decomposition of knot and link complements. The book of Eisenbud and Neumann [10] gives a detailed analysis of the structure of the JSJ-decomposition of links in homology spheres whose complements are graph manifolds. Our paper differs from their book in that we study the class of links in S^3 with no restriction on the complements. The aspect of this paper which is new is that the complicating factor is the Brunnian properties of the resulting companion links.

A once frequently quoted yet unpublished manuscript of Bonahon and Siebenmann [1] also investigates JSJ-decompositions of link complements in S^3. Various results on the $\mathbf{Z}_2$-equivariant JSJ-decomposition from the manuscript of Bonahon and Siebenmann appear in the survey of Kawauchi [22]. This is the part of their work that relates to Conway spheres. The current exposition is most closely related to the part of the Bonahon-Siebenmann manuscript [1] that does not appear in Kawauchi's survey. This paper in part duplicates the results of Schubert, Eisenbud, Neumann, Bonahon and Siebenmann, and we attempt to give full credit to their discoveries.

This article started out as a technical lemma needed to determine the class of hyperbolic 3-manifolds that appear as components of a knot complement split along its JSJ-tori. The answer, although known to some, is not 'well known', which motivated the author to put together the present exposition. I would like to thank Allen Hatcher for several early suggestions on how to approach the topic. I'd also like to thank Gregor Masbaum for his suggestion of reformulating what is now Proposition 1, which led to the connection with Schubert's paper [30]. I'd like to thank Daniel Moskovich and David Cimasoni for their comments on the paper, and Francis Bonahon for encouragement.

2. Disjoint knot complements, and companions

This section starts with a technical proposition about disjoint knot complements in S^3 which ultimately motivates the notions of splicing and companions of a knot or link.

Proposition 1. *Let $C_1, C_2, \ldots, C_n$ be n disjoint submanifolds of S^3 such that $K_i = \overline{S^3 \setminus C_i}$ is a non-trivially embedded solid torus in S^3 for all $i \in \{1,2,\ldots,n\}$. Then there exists n disjointly embedded 3-balls $B_1, B_2, \ldots, B_n \subset S^3$ such that $C_i \subset B_i$ for all $i \in \{1,2,\ldots,n\}$. Moreover, each B_i can be chosen to be C_i union a 2-handle which is a tubular neighbourhood of a meridional disc for K_i.*

Proof. For all $i \in \{1,2,\ldots,n\}$ let D_i^2 be a meridional disc for K_i.

Consider the case that we have j disjoint 3-balls $B_1, B_2, \ldots, B_j$ such that $C_i \subset B_i$ for all $i \in \{1,2,\ldots,j\}$ with B_i disjoint from C_l for all $i \neq l$, $i \in \{1,2,\ldots,j\}$ and $l \in \{1,2,\ldots,n\}$. We proceed by induction, the base-case being the trivial $j=0$ case.

Consider the intersection of D_{j+1}^2 with $\partial B_1 \cup \partial B_2 \cup \cdots \cup \partial B_j \cup \partial C_{j+2} \cup \partial C_{j+3} \cup \cdots \cup \partial C_n$.

- If the intersection is empty, let B_{j+1} be a regular neighbourhood of $C_{j+1} \cup D_{j+1}^2$.
- If on the other hand the intersection is non-empty, let S be an innermost curve of the intersection bounding an innermost disc D in D_{j+1}^2. Thus S is a sub-manifold of one of $\partial B_1, \ldots, \partial B_j$ or $\partial C_{j+2}, \ldots, \partial C_n$.
  - If S bounds a disc D' in some ∂B_i for $i \leq j$ or ∂C_i for $j+2 \leq i \leq n$, then $D \cup D'$ bounds a ball in B_i or C_i respectively, which gives a natural isotopy of D_{j+1}^2 which lowers the number of components of intersection with the family $\partial B_1 \cup \partial B_2 \cup \cdots \cup \partial B_j \cup \partial C_{j+2} \cup \partial C_{j+3} \cup \cdots \cup \partial C_n$.
  - If S does not bound a disc in the above family, it must be a meridional curve in some ∂C_i for $j+2 \leq i \leq n$. In this case, we let B_i be a regular neighbourhood of $C_i \cup D$.

Thus by re-labelling the tori and balls appropriately, we have completed the inductive step.

Proposition 1 first appears in the literature as a theorem of Schubert [30] (§ 15.1, p. 199). It also appears in Sakuma's [28] work on the symmetry properties of knots. A related result was re-discovered by Bonahon and

Siebenmann in their unpublished manuscript [1] as part of their algorithm to determine if a 'splicing tree' results in the construction of a link embedded in S^3.

PROPOSITION 2. *Let M be a compact submanifold of S^3 with ∂M a disjoint union of n tori. By Alexander's theorem, $\overline{S^3 \setminus M}$ consists of a disjoint union of p solid tori and q non-trivial knot complements, where $p+q=n$. There exists an embedding $f\colon M \to S^3$ such that $f(M)$ is the complement of an open tubular neighbourhood of an n-component link $L \subset S^3$ which contains a q-component unlink as a sublink.*

Proof. Let $q \in \{0,1,\dots,n\}$ be the number of components of $\overline{S^3 \setminus M}$ which are non-trivial knot complements, and let $\overline{S^3 \setminus M} = C_1 \sqcup \cdots \sqcup C_n$ where C_i is a solid torus for $q+1 \le i \le n$ and a non-trivial knot-complement for $1 \le i \le q$. By Proposition 1 there exist disjoint 3-balls $B_1,\dots,B_q \subset S^3$ such that B_i is obtained from C_i by an embedded 2-handle attachment, $B_i = C_i \cup H_i^2$. Dually, C_i is obtained from B_i by drilling out a neighbourhood of a knotted properly-embedded interval.

Let $Q = \{(x,y,z) \in \mathbf{R}^3 : |(y,z)| \le \frac{1}{2}\} \cap D^3$. For $i \in \{1,2,\dots,q\}$ let $u_i\colon (D^3, A_i) \to (B_i, H_i^2)$ be an orientation-preserving diffeomorphism of pairs, where $A_i \subset D^3$ is a 3-ball such that $A_i \cap \partial D^3 = Q \cap \partial D^3$. For $i \in \{1,2,\dots,q\}$ let $w_i\colon Q \to A_i$ be a diffeomorphism which is the identity on $Q \cap \partial D^3 = A_i \cap \partial D^3$.

We define an embedding $f\colon M \to S^3$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \in \overline{M \setminus \bigcup_{i=1}^q H_i^2}, \\ u_i \circ w_i^{-1} \circ u_i^{-1}(x) & \text{if } x \in H_i^2. \end{cases}$$

By design, $f(\bigcup_{i=1}^q \partial C_i)$ bounds a tubular neighbourhood of a q-component unlink in the complement of $f(M)$. To argue that $f(M)$ is a link complement, notice that in our definition f extends naturally to an embedding $\overline{S^3 \setminus \bigcup_{i=1}^q C_i} \to S^3$.

DEFINITION 1. Let $M \subset S^3$ be a 3-manifold, and let $T \subset \partial M$ be a torus. Provided C is the component of $\overline{S^3 \setminus M}$ containing T, an essential curve $c \subset T$ is called an external (resp. internal) peripheral curve for M at T if $c = \partial S$ for some properly-embedded surface $S \subset C$ (resp. $S \subset \overline{S^3 \setminus C}$).

PROPOSITION 3. *Let $M \subset S^3$ be a 3-manifold whose boundary is a disjoint union of tori. Up to isotopy, there exists a unique orientation-preserving embedding $f\colon M \to S^3$ such that (1) and (2) are true:*

(1) *$f(M)$ is the complement of a tubular neighbourhood of a link in S^3.*

(2) *f maps external peripheral curves of ∂M to external peripheral curves of $\partial(f(M))$.*

f will be called the untwisted re-embedding of M.

Proof. We prove existence in the framework of the proof of Proposition 2. Let $Q' = \{(x,y,z) \in \partial D^3 : -\frac{\sqrt{3}}{4} \leq x \leq \frac{\sqrt{3}}{4}\}$. For each $i \in \{1,2,\dots,q\}$ extend w_i to be a homeomorphism $w_i\colon Q \cup Q' \to A_i \cup Q'$ so that $w_i(x) = x$ for all $x \in Q'$. Fix the curves $c_1 = \{(\cos\theta, \sin\theta, 0) : \pi \leq \theta \leq 2\pi\} \cup \{(x, 0, 0) : -1 \leq x \leq 1\}$ and $c_2 = \{(\cos\theta, \sin\theta, 0) : \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}\} \cup \{(x, \frac{1}{2}, 0) : -\frac{\sqrt{3}}{4} \leq x \leq \frac{\sqrt{3}}{4}\}$. For each $i \in \{1,2,\dots,q\}$ there is a unique choice of w_i up to isotopy so that $lk(w_i(c_1), w_i(c_2)) = 0$, since any two choices of w_i differ by some Dehn twist about a disc in Q separating the components of $Q \cap \partial D^3$. With this choice, then external peripheral curves are sent to external peripheral curves.

To prove uniqueness, let $f_1\colon M \to S^3$ and $f_2\colon M \to S^3$ be any two embeddings of M in S^3 as link complements sending external peripheral curves to external peripheral curves. Let $\overline{M}$ be the Dehn filling of M where the attaching maps are given by the external peripheral curves of M. Then f_1 and f_2 extend to orientation-preserving diffeomorphisms $\bar{f}_1, \bar{f}_2\colon \overline{M} \to S^3$. Cerf's theorem [8] states that any two orientation-preserving diffeomorphisms of S^3 are isotopic, thus f_1 and f_2 are isotopic.

DEFINITION 2. Given L a link in S^3 let $C_L = S^3 \setminus U_L$ where U_L is any open tubular neighbourhood of L. If $M \subset C_L$ is a manifold with incompressible torus boundary let $f\colon M \to S^3$ be its untwisted re-embedding. Then $f(M)$ is the complement of some link L' in S^3, unique up to unoriented isotopy. Any such link will be called a companion link to L. If M is a component of one of the prime summands of C_L split along the tori of its JSJ-decomposition, we call L' a JSJ-companion link to L. A link L will be called compound if C_L is reducible or if it is irreducible with a non-empty JSJ-decomposition.

Thus a link is non-compound if and only if its complement is prime and atoroidal.

3. Seifert-fibred submanifolds of S^3

In this section we determine which Seifert-fibred manifolds embed in S^3, and the various ways in which they embed. This allows us to classify the links in S^3 whose complements are Seifert-fibred, and give basic restrictions on which Seifert-fibred manifolds can be adjacent in the JSJ-decomposition of a 3-manifold in S^3.

Lemma 1. *If M is a sub-manifold of S^3 with non-empty boundary a union of tori, then either M is a solid torus or a component of $\overline{S^3 \setminus M}$ is a solid torus.*

Proof. Let $C = \overline{S^3 \setminus M}$. Since ∂M consists of a disjoint union of tori, every component of ∂M contains an essential curve α which bounds a disc D in S^3. Isotope D so that it intersects ∂M transversely in essential curves. Then $\partial M \cap D \subset D$ consists of a finite collection of circles, and these circles bound a nested collection of discs in D. Take an innermost disc D'. If $D' \subset M$ then M is a solid torus. If $D' \subset C$ then the component of C containing D' is a solid torus.

We will use the notation in Hatcher's notes [14] for describing orientable Seifert-fibred manifolds. In short, let $\mathcal{S}_{g,b}$ denote a compact surface of genus g with b boundary components. If $g < 0$, $\mathcal{S}_{g,b}$ is the connect-sum of $-g$ copies of $\mathbf{R}P^2$ with $\mathcal{S}_{0,b}$. $M(g,b\,;\frac{\alpha_1}{\beta_1},\frac{\alpha_2}{\beta_2},\ldots,\frac{\alpha_k}{\beta_k})$ denotes the Seifert-fibred manifold fibred over $\mathcal{S}_{g,b}$ with at most k singular fibres, and fibre-data α_i/β_i. One constructs $M(g,b\,;\frac{\alpha_1}{\beta_1},\frac{\alpha_2}{\beta_2},\ldots,\frac{\alpha_k}{\beta_k})$ from the orientable S^1-bundle over $\mathcal{S}_{g,b+k}$ by Dehn filling along k of the boundary components using the attaching slopes $\frac{\alpha_i}{\beta_i}$ for $i \in \{1,2,\ldots,k\}$.

Definition 3. We give a non-standard but flexible notation for defining unoriented isotopy classes of links in S^3 which are the union of fibres from a Seifert fibring of S^3. Provided $(p,q) \in \mathbf{Z}^2$ satisfies $p \neq 0$ and $q \neq 0$, with $X \subset \{*_1, *_2\}$, $S(p,q \mid X)$ denotes the subspace of S^3 made up of the union of three disjoint sets S_1, S_2, S_3 where:

- $S_1 = \{(z_1,z_2) \in S^3 \subset \mathbf{C}^2 : z_1^p = z_2^q\}$;
- $S_2 = \{(z_1,0) \in S^3\}$ provided $*_1 \in X$, otherwise $S_2 = \varnothing$;
- $S_3 = \{(0,z_2) \in S^3\}$ provided $*_2 \in X$, otherwise $S_3 = \varnothing$.

We mention some shorthand notation for some common links with Seifert-fibred complements. The Hopf link H^1 is the 2-component link in S^3 given by the union of S_2 and S_3 above, alternatively $H^1 = S(2,2 \mid \varnothing)$. Up to isotopy (modulo re-indexing) there are two Hopf links, distinguished by the linking number of the components (see Figure 1).

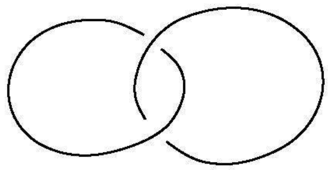

FIGURE 1

If one takes a connected-sum of p copies of the Hopf link, one obtains the $(p+1)$-component 'key-chain link' H^p. Modulo re-indexing, there are $p+1$ distinct $(p+1)$-component key-chain links. When orientation matters, we will use the 'right handed' key-chain link where all non-zero linking numbers are positive (see Figure 2).

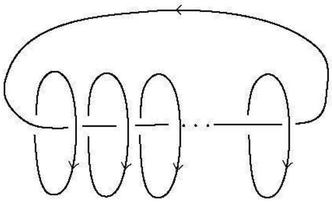

FIGURE 2

For any $(p,q) \in \mathbf{Z}^2$ with $p \nmid q$, $q \nmid p$ and $GCD(p,q) = 1$ the (p,q)-*torus knot* is $T^{(p,q)} = S(p,q \mid \varnothing)$. There is only one (p,q)-torus knot, since all torus knots are are invertible. The conditions $q \nmid p$ and $p \nmid q$ ensure that the unknot is not a torus knot.

For any $(p,q) \in \mathbf{Z} \times \mathbf{Z}$ with $p \nmid q$ and $GCD(p,q) = 1$, the (p,q)-*Seifert link* $S^{(p,q)}$ is defined to be $S^{(p,q)} = S(p,q \mid \{*_1\})$. We fix the orientation on $*_1$ counter-clockwise and orient the remaining component by the parametrisation $(\frac{z^q}{\sqrt[p]{2}}, \frac{z^p}{\sqrt[q]{2}})$ where $z \in S^1$. Our condition $p \nmid q$ is there to ensure that the Hopf link is not considered to be a Seifert link.

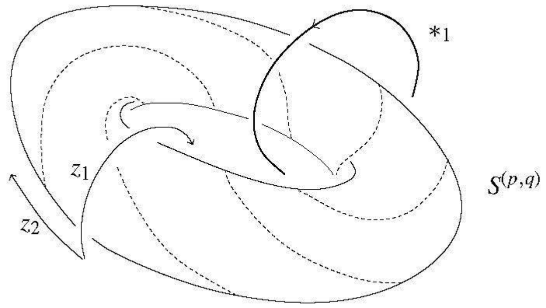

FIGURE 3

PROPOSITION 4. *Let $V \neq S^3$ be a Seifert-fibred sub-manifold of S^3, then V is diffeomorphic to one of the following:*

- $M(0,n;\frac{\alpha_1}{\beta_1},\frac{\alpha_2}{\beta_2})$ *for $n \geq 1$ and $\alpha_1\beta_2+\alpha_2\beta_1=\pm1$. These appear only as the complements of n regular fibres in a Seifert fibring of S^3.*
- $M(0,n;\frac{\alpha_1}{\beta_1})$ *for $n \geq 1$. These appear only as complements of $n-1$ regular fibres in a Seifert-fibring of an embedded solid torus in S^3.*
- $M(0,n;)$ *for $n \geq 2$. These appear in two different ways:*
  - *As complements of the singular fibre and $n-1$ regular fibres in a Seifert-fibring an embedded solid torus in S^3.*
  - *A manifold whose untwisted re-embedding in S^3 is the complement of a key-chain link.*

Proof. Consider $V \simeq M(g,b;\frac{\alpha_1}{\beta_1},\frac{\alpha_2}{\beta_2},\ldots,\frac{\alpha_k}{\beta_k})$. By design, $b=0$ if and only if $V=S^3 \simeq M(0,0;\frac{\alpha_1}{\beta_1},\frac{\alpha_2}{\beta_2})$ where $\alpha_1\beta_2+\alpha_2\beta_1=\pm1$.

Consider the case $b \geq 1$. Seifert-fibred manifolds that fibre over a non-orientable surface do not embed in S^3 since an orientation-reversing closed curve in the base lifts to a Klein bottle, which does not embed in S^3 by the Generalised Jordan Curve Theorem [13], thus $g \geq 0$.

A Seifert-fibred manifold that fibres over a surface of genus $g>0$ does not embed in S^3 since the base manifold contains two curves that intersect transversely at a point. If we lift one of these curves to a torus in S^3, it must be non-separating. This again contradicts the Generalised Jordan Curve Theorem, thus $g=0$.

By Lemma 1, either V is a solid torus $V \simeq M(0,1\,;\,\frac{\alpha_1}{\beta_1})$ or some component Y of $\overline{S^3 \setminus V}$ is a solid torus. Consider the latter case. There are two possibilities.

1. *The meridians of Y are fibres of V.* If there is a singular fibre in V, let β be an embedded arc in the base surface associated to the Seifert-fibring of V which starts at the singular point in the base and ends at the boundary component corresponding to ∂Y. β lifts to a 2-dimensional CW-complex in V, and the endpoint of β lifts to a meridian of Y, thus it bounds a disc. If we append this disc to the lift of β, we get a CW-complex X which consists of a 2-disc attached to a circle. The attaching map for the 2-cell is multiplication by β where $\frac{\alpha}{\beta}$ is the slope associated to the singular fibre. The boundary of a regular neighbourhood of X is a 2-sphere, so we have decomposed S^3 into a connected sum $S^3 = L_{\frac{\gamma}{\beta}} \# Z$ where $L_{\frac{\gamma}{\beta}}$ is a lens space with $H_1 L_{\frac{\gamma}{\beta}} = \mathbf{Z}_\beta$. Since S^3 is irreducible, $\beta = 1$. Thus $V \simeq M(0,n;)$ for some $n \geq 1$. Consider the untwisted re-embedding $f\colon V \to S^3$, and let $V' = f(V)$. V' is the complement of some link $L = (L_0, L_1, \ldots, L_{n-1})$ such that $(L_1, \ldots, L_{n-1})$ is an unlink (provided we let L_0 correspond to Y). C_{L_0} is obtained from V' by Dehn filling on $L_1, \ldots, L_{n-1}$ with integral slopes, thus C_{L_0} is a solid torus.
2. *The meridians of Y are not fibres of V.* In this case, we can extend the Seifert fibring of V to a Seifert fibring of $V \cup Y$. Either $V \cup Y = S^3$, or $V \cup Y$ has boundary.
   - If $V \cup Y = S^3$ then we know by the classification of Seifert fibrings of S^3 that any fibring of S^3 has at most two singular fibres. If V is the complement of a regular fibre of a Seifert fibring of S^3, then V is a torus knot complement $V \simeq M(0,1\,;\,\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ with $\alpha_1\beta_2 + \alpha_2\beta_1 = \pm 1$. Otherwise, V is the complement of a singular fibre, meaning that V is a solid torus $M(0,1\,;\,\frac{\alpha}{\beta})$.
   - If $V \cup Y$ has boundary, we can repeat the above argument. Either $V \cup Y$ is a solid torus, in which case $V \simeq M(0,2\,;\,\frac{\alpha_1}{\beta_1})$, or a component of $S^3 \setminus \overline{V \cup Y}$ is a solid torus, so we obtain V from the above manifolds by removing a Seifert fibre. By induction, we obtain V from either a Seifert fibring of a solid torus, or a Seifert fibring of S^3 by removing fibres. Thus either $V \simeq M(0,n\,;\,\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ for $n \geq 1$ with $\alpha_1\beta_2 + \alpha_2\beta_1 = \pm 1$, $V \simeq M(0,n\,;\,\frac{\alpha_1}{\beta_1})$ for $n \geq 1$, or $V \simeq M(0,n;)$ for $n \geq 2$. In order, these are the cases where we remove only regular fibres from a fibring of S^3, regular fibres from a fibring of a solid torus, and regular fibres plus the singular fibre from a fibring of a solid torus.

PROPOSITION 5. *Each link in S^3 whose complement admits a Seifert-fibring is isotopic to some $S(p,q \mid X)$, excepting only the key-chain links.*

Given $p,q \in \mathbf{Z} \setminus \{0\}$ let $p' = p/GCD(p,q)$, $q' = q/GCD(p,q)$ and let $m,l \in \mathbf{Z}$ satisfy $p'm + lq' = 1$, then the complement of $S(p,q \mid X)$ is diffeomorphic to:

- $M(0, GCD(p,q)\,; \frac{m}{q'}, \frac{l}{p'})$ *provided* $X = \varnothing$;
- $M(0, 1+GCD(p,q)\,; \frac{m}{q'})$ *provided* $X = \{*_1\}$;
- $M(0, 1+GCD(p,q)\,; \frac{l}{p'})$ *provided* $X = \{*_2\}$;
- $M(0, 2+GCD(p,q)\,;)$ *provided* $X = \{*_1, *_2\}$.

The complement of the key-chain link H^p is diffeomorphic to $M(0, p+1;)$.

Let A denote an index-set for the components of $S(p,q \mid X)$ that are neither $*_1$ nor $*_2$. Let X' be the collection of singleton subsets of X. Then the strong Brunnian property of $S(p,q \mid X)$ is given by:

- $\{\{*_1\}\}$ *for the unknot* $S(1,1|)$;
- $\{\{*_1\},\{*_2\}\}$ *for the Hopf link* $H^1 = S(2,2|)$;
- $\{\{a\} : a \in A\} \cup X'$ *if* $p \mid q$ *or* $q \mid p$;
- X' *if* $p \nmid q$ *and* $q \nmid p$.

Proof. Except for the Brunnian properties, this result follows immediately from Proposition 4. To see the Brunnian properties, observe that the linking number between regular fibres of $S(p,q \mid X)$ is $p'q'$, the linking number between $*_1$ and $*_2$ is one, and the linking number between a regular fibre and either $*_1$ or $*_2$ is p' or q' respectively.

Proposition 5 first appears in the literature in the paper of Burde and Murasugi [6], where they classify links in S^3 whose complements are Seifert-fibred.

COROLLARY 1. *Let L be a link in S^3 such that C_L admits a Seifert fibring. Provided L is not the unknot or the Hopf link, the fibring is unique up to isotopy.*

Proof. $S(p,q \mid X)$ is the unknot if and only if $X = \varnothing$, $GCD(p,q) = 1$ and either $p = \pm 1$ or $q = \pm 1$. $S(p,q) \mid X)$ is the Hopf link if and only if either $X = \varnothing$ and $p = \pm q = \pm 2$ or $|X| = 1$ with $p = \pm q = \pm 1$. Thus, the complements of $S(p,q \mid X)$ which are not unknot complements or Hopf link complements all have the form $M(0, b\,; \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$ where the sum of the number of boundary components plus the number of singular fibres is at least 3 with

$\alpha_1\beta_2 + \alpha_2\beta_1 = \pm 1$. That these manifolds have unique Seifert-fibrings up to orientation-preserving diffeomorphism follows from Theorem 2.3 in Hatcher's notes [14]. Consider a horizontal essential annulus S in one of these manifolds. We have the basic relation among Euler characteristics $\chi(B) - \frac{\chi(S)}{n} = \sum_i (1 - \frac{1}{\beta_i})$ where B is the base space. Since $\chi(S) = 0$, this equation has no solution. Thus by Proposition 1.11 of [14] all essential annuli are vertical, therefore any diffeomorphism of these manifolds is fibre-preserving.

PROPOSITION 6. *Let θ be the unique involution of the set $\{*_1, *_2\}$. Let $\mathcal{C} = \{S(p,q \mid X) : (p,q) \in (\mathbf{Z} \setminus \{0\})^2, X \subset \{*_1, *_2\}\}$. The equivalence relation $\sim$ of unoriented isotopy on $\mathcal{C}$ is generated by the relations:*

(1) $S(p,q \mid X) \sim S(-p,-q \mid X) \quad \forall (p,q) \in \left(\mathbf{Z} \setminus \{0\}\right)^2,\ X \subset \{*_1, *_2\}$.

(2) $S(p,q \mid X) \sim S(q,p \mid \theta(X)) \quad \forall (p,q) \in \left(\mathbf{Z} \setminus \{0\}\right)^2,\ X \subset \{*_1, *_2\}$.

(3) $S(p,q \mid X \cup \{*_1\}) \sim S(p + \frac{p}{q}, q+1 \mid X \setminus \{*_1\}) \quad \forall q \mid p$ *with* $q > 0$, $X \subset \{*_1, *_2\}$.

(4) $S(p,q \mid X) \sim S(-p,q \mid X)$ *if either (a)* $X = \varnothing$, $GCD(p,q) = 1$ *with* $p = \pm 1$ *or* $q = \pm 1$, *(b)* $X = \varnothing$, $p,q = \pm 2$ *or (c)* $|X| = 1$, $\pm p = \pm q = 1$. *This is the case where $S(p,q \mid X)$ is an unknot or Hopf link.*

Proof. Restrict to the sub-class of $\mathcal{C}$ consisting of $S(p,q \mid X)$ which are not unknot or Hopf links. These complements have a unique Seifert-fibring by Proposition 1. If $X = \varnothing$, $p \nmid q$ and $q \nmid p$, observe that items (1) and (2) generate $\sim$, this is because the classification of Seifert-fibred spaces up to fibre-equivalence (Proposition 2.1 in [14]) tells us that $\sim$ is equivalent to the fibre-equivalence relation, thus we have proven more: up to isotopy there is only one orientation-preserving embedding of the complement of $S(p,q \mid \varnothing)$ in S^3. If we broaden the class to include X non-empty, (1) and (2) still suffice to generate $\sim$ essentially because $S(p,q \mid \varnothing)$ is contained as a sublink. Consider a general $S(p,q \mid X)$. Let $p' = p/GCD(p,q)$ and $q' = q/GCD(p,q)$. We know p', q' and $p'q'$ are the possible linking numbers of the components of $S(p,q \mid X)$ thus we can determine whether or not $p = \pm q$ via linking numbers. For such a link, the relative sign $p/q \in \{\pm 1\}$ can be computed by coherently orienting three strands and computing the linking number of two of them. Thus such links are classified by the number of their components together with the sign $p/q \in \{\pm 1\}$ which is equivalent to relations (1), (2) and (3). Consider the case $q \mid p$ but $p \nmid q$. In this case $*_2 \in S(p,q \mid X)$ if and only some linking number is ± 1, and the linking number with $*_2$ would be q'. Thus each such is equivalent via relations

(1) through (3) to some $S(q, p'q \mid X)$ where X is either empty or contains $*_2$, $q > 0$ and $|p'| > 1$. q is the number of components of $S(q, p'q \mid X)$. Since we have assumed $S(q, p'q \mid X)$ is not the unknot nor the Hopf link, $q + |X| \geq 3$ thus we can compute p' as a linking number of two coherently oriented strands of $S(q, p'q \mid X)$ either in the complement of $*_2$ if $q = 2$ or in the complement of another strand, thus relations (1) through (3) suffice.

In the exceptional case of the unknot or the Hopf link, $p/q \in \{\pm 1\}$ is not an invariant, thus the exceptional relation (4).

DEFINITION 4. Given a Seifert-fibred 3-manifold $M \subset S^3$ with $T \subset \partial M$, a boundary component, the *fibre-slope of M at T* is $\frac{\alpha}{\beta}$ provided $\pm(\alpha c_e + \beta c_i) \in H_1(T; \mathbf{Z})$ represents the homology class of a Seifert fibre, where $c_e, c_i \subset T$ are external and internal peripheral curves for T such that $lk(c'_e, c_i) = +1$ where $c'_e \subset int(M)$ is parallel to c_e.

Given a link L indexed by a set A with C_L Seifert-fibred, for $a \in A$ define the *a-th fibre-slope of L* to be the fibre-slope of C_L at the boundary torus corresponding to L_a.

The proof of the following is immediate.

PROPOSITION 7. *If $M \subset S^3$ is Seifert-fibred with $T \subset \partial M$ a boundary torus and $f: M \to S^3$ the untwisted re-embedding of M, then the fibre-slope of M at T is the fibre-slope of $f(M)$ at $f(T)$. Moreover, if $M \cup N \subset S^3$ is Seifert-fibred with $M \cap N = T$, then the fibre-slope of M at T is the inverse of the fibre-slope of N at T.*

The importance of Proposition 7 is that it gives us an obstruction to two Seifert-fibred manifolds being adjacent in a link complement split along its JSJ-tori.

PROPOSITION 8. *The fibre-slopes of $S(p, q \mid X)$ are $LCM(p, q)/GCD(p, q)$ for regular fibres, p/q for $*_1$ and q/p for $*_2$.*

The Thurston geometries [33] on the Seifert-fibred submanifolds of S^3 turn out to be non-unique. We give a sketch of how to construct them. Milnor [25] has shown that any Seifert-fibred link complement in S^3 is the total space of a fibre bundle over S^1. If M is Seifert-fibred and $f: M \to S^1$ is a fibre-bundle, let $F = f^{-1}(1)$ be the fibre. The monodromy (classifying/gluing map), as an element of the mapping-class group of F, is always of finite order provided

F is not a torus. This is because F is essential in M so we can assume that either F is a union of Seifert-fibres, or it is transverse everywhere to the Seifert-fibres [14]. If F is transverse to the Seifert-fibres then as an element of the mapping class group of F ($\pi_0\mathrm{Diff}(F)$) the monodromy has order equal to $|F \cap c|$ provided c is a regular fibre. If F is a union of fibres, it is either $S^1 \times [0,1]$ or a torus, and the mapping class group of $S^1 \times [0,1]$ is finite. The interior of $\mathcal{S}_{g,b}$ is hyperbolic for $2g+b \geq 3$, thus the monodromy is an isometry of the fibre for a suitable hyperbolic metric on the fibre [23]. Of course, the only link complements that fibre over S^1 with $F \simeq \mathcal{S}_{g,b}$ satisfying $2g+b<3$ are the Hopf link and the unknot. Theorem 4.7.10 of [33] tells us that on top of having the above finite-volume $\mathbf{H}^2 \times \mathbf{E}^1$-structure, Seifert-fibred link complements also have an $\widetilde{PSL}(2,\mathbf{R})$-structure.

Here is an example of how one can find the $\mathbf{H}^2 \times \mathbf{E}^1$-structures on the complement of a torus knot. Let $C_{p,q} = S^3 \setminus T^{(p,q)}$. Think of $T^{(p,q)}$ as the roots of the polynomial $f(z_1,z_2) = z_1^p - z_2^q$ where $(z_1,z_2) \in S^3 \subset \mathbf{C}^2$. The fibring $g\colon C_{p,q} \to S^1$ is given by $g(z_1,z_2) = \frac{f(z_1,z_2)}{|f(z_1,z_2)|}$. Define $C'_{p,q} = \mathbf{C}^2 \setminus f^{-1}(0)$. There is an action of the positive reals $\mathbf{R}^+$ on $C'_{p,q}$ given by $t.(z_1,z_2) = (t^{\frac{1}{p}}z_1, t^{\frac{1}{q}}z_2)$. As a function of t, $|t.(z_1,z_2)|$ is strictly increasing, thus $C'_{p,q} \simeq \mathbf{R}^+ \times C_{p,q}$. The function $f\colon C'_{p,q} \to \mathbf{C} \setminus \{0\}$ is a submersion thus a locally-trivial fibre bundle. If we let $C''_{p,q} = f^{-1}(S^1) \subset C'_{p,q}$ then $\mathbf{R}^+ \times C''_{p,q} \simeq C'_{p,q}$ since $\mathbf{C} \setminus \{0\} \simeq \mathbf{R}^+ \times S^1$. So $C''_{p,q}$ and $C_{p,q}$ are homotopy-equivalent. Moreover, define $g'\colon C'_{p,q} \to S^1$ and $g''\colon C''_{p,q} \to S^1$ by $g'(z_1,z_2) = \frac{f(z_1,z_2)}{|f(z_1,z_2)|}$ with g'' the restriction of g'. This makes the homotopy-equivalence $C''_{p,q} \to C_{p,q}$ a fibre homotopy-equivalence from g'' to g. Since both surfaces are 1-ended, they are diffeomorphic, moreover the homotopy-equivalence preserves the peripheral structures of the fibres, so g and g'' are smoothly-equivalent bundles. The fibre of g'', $g''^{-1}(1)$ is $\{(z_1,z_2) \in \mathbf{C}^2 : z_1^p - z_2^q = 1\}$ so the projection map $\pi\colon \mathbf{C}^2 \to \mathbf{C}$ given by $\pi(z_1,z_2) = z_2$ restricts to a p-sheeted branched covering space with $g''^{-1}(1)$ as the total space, $\mathbf{C}$ the base space, having q branch points in $\mathbf{C}$, all with ramification number p. This makes $g''^{-1}(1)$ a surface of genus $(p-1)(q-1)/2$. The monodromy $g''^{-1}(1) \to g''^{-1}(1)$ can be written explicitly as the map $(z_1,z_2) \longmapsto (e^{2\pi i/p}z_1, e^{2\pi i/q}z_2)$. As a covering transformation of $g''^{-1}(1)$ it is of order pq, with $p+q$ points of ramification, p of which have ramification number q, q having ramification number p, branched over a disc with two marked points. The idea for this computation comes from an example in an appendix of Paul Norbury to the notes of Walter Neumann [27].

Other than Seifert-fibred manifolds, a primary source for atoroidal manifolds is hyperbolic manifolds. Given an incompressible torus T in a complete hyperbolic 3-manifold M, then one would have an injection

$$\pi_1 T \to \pi_1 M \subset \mathrm{Isom}(\mathbf{H}^3)$$

where $\mathrm{Isom}(\mathbf{H}^3)$ is the group of isometries of hyperbolic 3-space. By the classification of hyperbolic isometries (see for example Proposition 2.5.17 in Thurston's book [33]) any subgroup of $\mathrm{Isom}(\mathbf{H}^3)$ isomorphic to $\mathbf{Z}^2$ consists entirely of parabolic elements. The Margulis Lemma, when applied to $\pi_1 T \to \mathrm{Isom}(\mathbf{H}^3)$ (see for example [34] or [21]) tells us that $M | T$ consists of two manifolds, one of which is diffeomorphic to $T \times [0, \infty)$, thus M is atoroidal. Thurston went on to prove a rather sharp converse [32]: the interior of a compact 3-manifold M with non-empty boundary admits a complete hyperbolic metric if and only if M is prime, 'homotopically atoroidal' and not the orientable I-bundle over a Klein bottle, moreover the metric is of finite volume if and only if ∂M is a disjoint union of tori. 'Homotopically atoroidal' means that subgroups of $\pi_1 M$ isomorphic to $\mathbf{Z}^2$ are peripheral. A convenient refinement is that if M is a compact, irreducible, atoroidal, 3-manifold with incompressible boundary containing no essential annulus with $T \subset \partial M$ the torus boundary components of M, then $M \setminus T$ admits a unique complete hyperbolic metric of finite volume with totally-geodesic boundary [21, 2, 34]. If $\partial M = \varnothing$ then M is also known to be hyperbolic provided M contains an incompressible surface. By uniqueness, the hyperbolic geometry is a topological invariant and it can be used to distinguish isotopy-classes of knots and links. There is a corresponding theory of orbifolds that allows one to go further and get strong geometric invariants of smoothly-embedded graphs in S^3 [15].

4. Companionship graphs for knots and links, splicing

In this section we define two graphs associated to a link. The first, called the 'JSJ-graph' (Definition 5), describes the basic structure of the JSJ-decomposition of a link's complement. We decorate the vertices of the JSJ-graph to get the 'companionship graph' of a link (Definition 6), which, in Proposition 9 we show is a complete isotopy invariant of links. We turn our attention first to knots. Companionship graphs for knots have the particularly simple form of a rooted tree with vertices labelled by '*knot generating links*' (Definition 7). We develop a notion of splicing for knots in

Definition 10. This allows us to inductively construct knots with prescribed companionship graphs. We describe the basic combinatorics of companionship graphs for knots under splicing in Proposition 14. Theorem 2 gives a complete characterisation of companionship graphs for knots, after which we give various examples.

We then turn our attention to companionship graphs of links. We define the notion of a 'splice diagram' in Definition 8. A splice diagram is not a concept of any real importance of its own, as it simply codifies some of the most elementary properties of a companionship graph. The 'Local Brunnian Property' (Proposition 15) is the first fundamental property of companionship graphs. We show in Proposition 16 that splice diagrams satisfying the Local Brunnian Property essentially encode for collections of disjoint embedded tori in link complements. We proceed to call these diagrams 'valid'. In Definition 14 we give a revised notion of splicing suitable for links. The 'fibre-slope exclusion property' (Lemma 16) is the other main property satisfied by companionship graphs, as it encodes the minimality condition of the JSJ-decomposition. Proposition 18 gives a complete characterisation of graphs that arise as companionship graphs of links: they are the valid splice diagrams, labelled by Seifert-fibred and hyperbolic links satisfying the fibre-slope exclusion property. In Proposition 19 we describe how companionship graphs behave under splicing, the most complicated case being the case of splicing with the unknot.

DEFINITION 5. Given a topological space X, if $\sim$ denotes the equivalence relation « $x \sim y \Longleftrightarrow$ *there is a path from* x *to* y », define $[x] = \{y \in X : y \sim x\}$ and $\pi_0 X = \{[x] : x \in X\}$.

Given a non-split link L in S^3 indexed by a set A, let T be the JSJ-decomposition of C_L, indexed by a set B disjoint from A. The graph G_L is defined to have vertex-set $\pi_0(C_L \,|\, T)$ and edge set $\pi_0 T$. We give G_L the structure of a partially-directed graph, in that some edges will have orientation. Given $b \in B$ let M and N be the two components of $C_L \,|\, T$ containing T_b. If T_b bounds a solid torus W in S^3 on only one side, and if $M \subset W$, then we orient the edge $\pi_0 T_b$ so that its terminal point is $\pi_0 M$.

For a split link L with index set A, partition A as $A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_j$ so that $C_L = C_{L_{A_1}} \# \cdots \# C_{L_{A_j}}$ is the prime decomposition of C_L. Define G_L for a split link L to be $\bigsqcup_{i=1}^{j} G_{L_{A_i}}$. G_L is called the JSJ-graph of L (see Figures 4 and 5).

EXAMPLE 1. An example of a knot K and its JSJ-decomposition $T = \{T_1, T_2, T_3, T_4\}$. K is a connected sum of a trefoil, a figure-8 and the Whitehead double of a figure-8 knot.

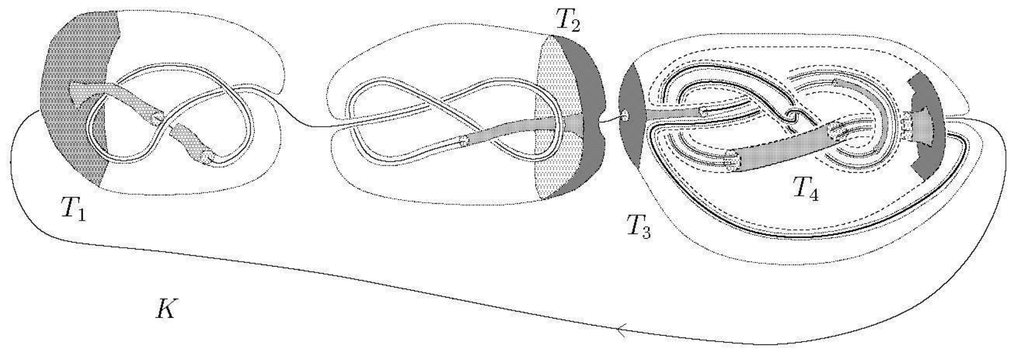

FIGURE 4

Notice that the component of $C_K \,|\, T$ containing ∂C_K has as its untwisted re-embedding the complement of H^3. T_1 bounds a trefoil complement, T_2 a figure-8 complement, T_3 the Whitehead double of the figure-8 knot and $T_3 \cup T_4$ bounds a manifold which, when re-embedded is the complement of the Whitehead link. The Whitehead link complement and figure-8 complement are atoroidal since they are finite-volume hyperbolic manifolds as mentioned at the end of Section 3.

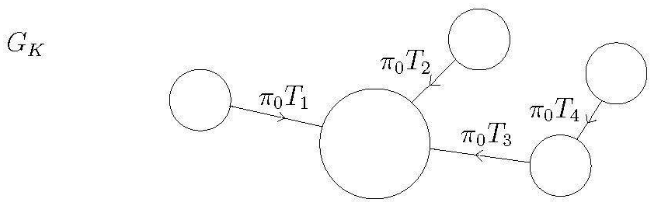

FIGURE 5

Graph constructions from the JSJ-decompositions of 3-manifolds were perhaps first made by Siebenmann [31] in the context of homology spheres. Eisenbud and Neuman [10] made similar constructions for knots in homology spheres whose complements are graph-manifolds.

DEFINITION 6. Given a non-split link L with index-set A, if T is the JSJ-decomposition of C_L, for each $v \in G_L$ let $M(v) \in C_L \,|\, T$ be the component corresponding to v. We define $\mathbf{G}_L$ to be the partially-directed graph such that each vertex v is labelled by a link $\mathbf{G}_L(v)$ satisfying:

1. If we forget the vertex labelling, $\mathbf{G}_L$ is the JSJ-graph G_L.
2. The unoriented isotopy class of $\mathbf{G}_L(v)$ is the companion link to the component $M(v)$.
3. If $A(v)$ denotes the subset of A corresponding to the components of $M(v) \cap \partial C_L$, and $E(v)$ the subset of edges of G_L incident to v, then $\mathbf{G}_L(v)$ is naturally indexed by the set $A(v) \sqcup E(v)$.
4. Given v_1 and v_2 adjacent vertices of $\mathbf{G}_L$, let $\{e\} = E(v_1) \cap E(v_2)$, then the orientation of $\mathbf{G}_L(v_1)_e$ and $\mathbf{G}_L(v_2)_e$ is chosen so that if f_1 and $\widetilde{f_2}$ are the untwisted re-embeddings $f_i \colon M(v_i) \to C_{\mathbf{G}_L(v_i)}$, then $lk(f_1^{-1}(l_1), \widetilde{f_2^{-1}}(l_2)) = +1$ where $l_i \subset \partial C_{\mathbf{G}_L(v_i)}$ is the standard longitude corresponding to $\mathbf{G}_L(v_i)_e$ respectively, and $\widetilde{f_2^{-1}}(l_2) \subset int(M(v_1))$ is a parallel translate of $f_2^{-1}(l_2)$.

If L is a split link, define $\mathbf{G}_L = \bigsqcup_{i=1}^k \mathbf{G}_{L_{A_i}}$ where $C_L \simeq C_{L_{A_1}} \# \cdots \# C_{L_{A_k}}$ is the prime decomposition of C_L. (Compare with Figures 5 and 6.)

Thus the union of the index sets for the links $\{\mathbf{G}_L(v) : v \in \mathbf{G}_L\}$ consist of A together with the edges of G_L. Counting with multiplicity, exactly one component of the links $\{\mathbf{G}_L(v) : v \in \mathbf{G}_L\}$ is indexed by any element of A, and precisely two (corresponding to adjacent vertices) are labelled by an edge of G_L. Components decorated by elements of A are called 'externally-labelled'. Components labelled by the edge-set of G_L are 'internally labelled'. If K is a knot, we consider it to be a link with index-set $\{*\}$.

Given an edge e of $\mathbf{G}_L$ with endpoints v_1 and v_2, if we reverse both of the orientations of $\mathbf{G}_L(v_1)_e$ and $\mathbf{G}_L(v_2)_e$, this would also satisfy the above definition, and $\mathbf{G}_L$ is well defined modulo this choice.

DEFINITION 7. A *KGL indexed by* (A, b) is a link L with index-set $A \sqcup \{b\}$ such that L_A is an unlink. When it does not cause confusion, we will frequently let $b = 0$ and $A = \{1, 2, \ldots, n\}$.

Some elementary observations one can make about $\mathbf{G}_L$:

- $\mathbf{G}_L$ is connected if and only if L is not split.

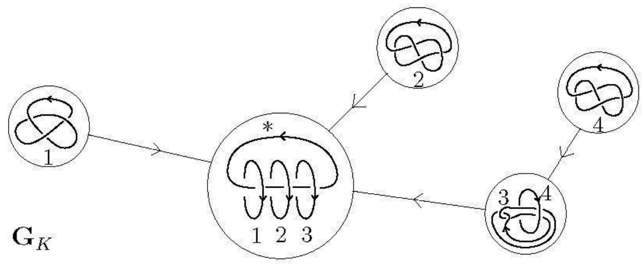

FIGURE 6

- $\mathbf{G}_L$ is acyclic, i.e. each component is a tree. This follows from the Generalised Jordan Curve Theorem [13], since an embedded torus in S^3 separates.
- If K is a knot then $\mathbf{G}_K$ is a rooted tree, since only one component of $C_K \,|\, T$ contains ∂C_K. By Proposition 2, for each vertex $v \in \mathbf{G}_K$, $\mathbf{G}_K(v)$ is a KGL. If we let $\mathbf{G}_K(v)$ be indexed by (A, b), then the edges of $\mathbf{G}_K$ corresponding to A terminate at v. Provided v is not the root of $\mathbf{G}_K$, b corresponds to an edge that starts at v and terminates at its parent. Thus, all the edges of $\mathbf{G}_K$ are oriented and all sufficiently-long directed paths in $\mathbf{G}_K$ terminate at the root.

DEFINITION 8. Given a finite set A, a *splicing diagram with external labels A* is an acyclic, partially-directed graph $\mathbf{G}$ such that each vertex $v \in \mathbf{G}$ is labelled by a link $\mathbf{G}(v)$ whose index-set is a subset of (the edge-set of $\mathbf{G}$)$\cup A$. We demand that if $e \in \mathbf{G}$ is an edge with $v_1, v_2 \in \mathbf{G}$ its endpoints, then one component of both $\mathbf{G}(v_1)$ and $\mathbf{G}(v_2)$ is labelled by e. These components of $\mathbf{G}(v_1)$ and $\mathbf{G}(v_2)$ are called 'internally-labelled'. We demand that for each $a \in A$ there exists a unique $v \in \mathbf{G}$ such that $\mathbf{G}(v)$ has a component indexed by a. We denote this vertex by v_a, and we say $\mathbf{G}(v_a)_a$ is 'externally-labelled'.

Given splicing diagrams $\mathbf{G}$ and $\mathbf{G}'$ with external index-sets A and A' respectively, we say $\mathbf{G}$ and $\mathbf{G}'$ are equivalent ($\mathbf{G} \sim \mathbf{G}'$) if $A = A'$ and there exists an isomorphism of partially-directed graphs $g\colon \mathbf{G} \to \mathbf{G}'$ together with unoriented isotopies $f(v)$ from $\mathbf{G}(v)$ to $\mathbf{G}'(g(v))$ for all $v \in \mathbf{G}$ such that:

- Given $a \in A$ and a component $\mathbf{G}(v)_a$, then $f(v)(\mathbf{G}(v)_a) = \mathbf{G}'(g(v))_a$.
- If $e \in \mathbf{G}$, $e' = g(e) \in \mathbf{G}$, with $v_1, v_2 \in \mathbf{G}$ the endpoints of e, and $v_i' = g(v_i)$ then $f(v_i)(\mathbf{G}(v_i)_e) = \epsilon\mathbf{G}'(v_i')_{e'}$ for some $\epsilon \in \{+, -\}$ which does not depend on $i \in \{1, 2\}$.

See the conventions regarding unoriented isotopies in Section 1 to make sense of the above definition.

PROPOSITION 9. *Two links L and Y are isotopic if and only if $\mathbf{G}_L \sim \mathbf{G}_Y$.*

Proof. '$\Longrightarrow$' is immediate since $\sim$ is an equivalence relation.

'$\Longleftarrow$' Let $h\colon S^3 \to S^3$ be an isotopy from L to Y. Thus $A = A'$ and $h(L_a) = Y_a$ for all $a \in A$, moreover $h(C_L) = C_Y$. Since the JSJ-decomposition is unique up to isotopy, then if T is the JSJ-decomposition of C_L, we can assume $h(T) \subset C_Y$ is the JSJ-decomposition of C_Y. If we let $M \in C_L \,|\, T$ then $h(M) \in C_Y \,|\, h(T)$ is isotopic to M, thus by Proposition 3, the companion link of C_L corresponding to M is unoriented isotopic to the companion link of $C_Y \,|\, h(T)$, thus we can let f be this isotopy.

The notion of 'splicing' was first described by Siebenmann [31] in his work on JSJ-decompositions of homology spheres. It was later adapted to the context of links in homology spheres by Eisenbud and Neumann [10]. We give a further refinement of splicing, adapted specifically so that it constructs knots in S^3.

DEFINITION 9. A *long knot* is an embedding $f\colon \mathbf{R} \times D^2 \to \mathbf{R} \times D^2$ satisfying:

- $supp(f) \subset [-1, 1] \times D^2$, i.e. f is the identity outside of $[-1, 1] \times D^2$.
- The linking number of $f_{|\mathbf{R}\times\{(0,0)\}}$ and $f_{|\mathbf{R}\times\{(1,0)\}}$ is zero.

From a long knot f, one can construct a knot in S^3 in a canonical way. The image of $f_{|\mathbf{R}\times\{(0,0)\}}$ is standard outside of $[-1, 1] \times D^2$ so its one-point compactification is a knot in $S^3 \equiv \dot{\mathbf{R}}^3$. This gives a bijective correspondence between isotopy classes of long knots, and isotopy classes of knots. The proof of this appears in many places in the literature ([5, 4] are recent examples) and essentially amounts to the observation that the unit tangent bundle to S^3 is simply-connected.

DEFINITION 10. Let $\mathbf{I} = [-\infty, \infty]$, and let $L = (L_0, L_1, \ldots, L_n)$ be an $(n+1)$-component KGL. Let $\widetilde{L}$ be a closed tubular neighbourhood of L. Let $C_{L_i} = \overline{S^3 \setminus \widetilde{L}_i}$ and $C_u = \cap_{i=1}^n C_{L_i}$. Let $h = (h_1, h_2, \ldots, h_n)$ be a collection of disjoint orientation-preserving embeddings $h_i\colon \mathbf{I} \times D^2 \to C_{L_i}$ such that $img(h_i) \cap \partial C_u = img(h_{i|\mathbf{I}\times\partial D^2})$ with $h_i(\{0\} \times S^1)$ an oriented longitude for L_i. We call h a *disc-system* for $\widetilde{L}$.

Given $J = (J_1, J_2, \ldots, J_n)$ an n-tuple of non-trivial knots in S^3, let $f = (f_1, f_2, \ldots, f_n)$ be their associated long knots. The re-embedding function associated to L, the disc-system h and knots J is an embedding $R_h[L, J]\colon C_u \to S^3$ defined by:

$$R_h[L,J] = \begin{cases} (h_i \circ f_i \circ h_i^{-1})(x) & \text{if } x \in img(h_i)\,, \\ x & \text{if } x \in \overline{C_u \setminus \bigsqcup_{i=1}^{n} img(h_i)}\,. \end{cases}$$

The splice of J along L is defined as

$$J \bowtie L = R_h[L, J](L_0).$$

EXAMPLE 2. If we use the figure-8 knot for J_1 and the knot 6_3 from Rolfsen's knot table for J_2, with L the Borromean Rings, then $K = J \bowtie L$ is as illustrated in Figure 7.

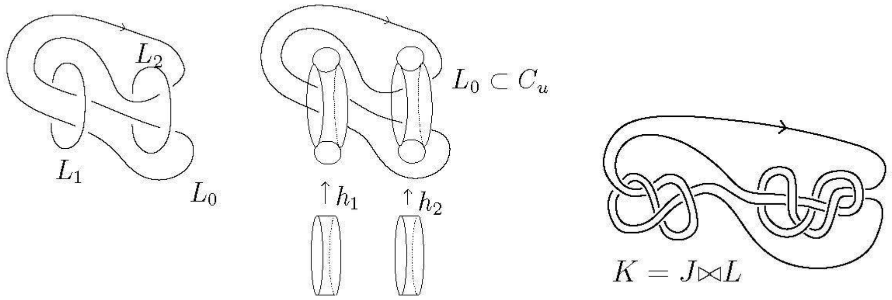

FIGURE 7

We will show that the isotopy-class of $R_h[L, J]$ does not depend on h. To do this, we show that any two disc systems are related by a finite sequence of 'elementary moves', and that disc systems related by a single elementary move give rise to isotopic re-embedding functions.

DEFINITION 11. Let h and h' be disc systems for $\widetilde{L}$. An *elementary move* on disc j from h to h' is a 1-parameter family of embeddings $H_i(t)\colon \mathbf{I} \times D^2 \to \overline{S^3 \setminus \widetilde{L}_i}$, $i \in \{1, 2, \ldots, n\}$, $t \in [0, 1]$, such that:

- $H_i(0) = h_i$ and $H_i(1) = h'_i$ for all $i \in \{1, \ldots, n\}$;
- $img(H_i(t)) \cap \partial\widetilde{L}_i = img(H_i(t)_{|\mathbf{I} \times \partial D^2})$ for all $i \in \{1, \ldots, n\}, t \in [0, 1]$;
- $H_i(t) = H_i(0)$ for all $i \neq j$ and $t \in [0, 1]$.

PROPOSITION 10. *If two disc-systems h and h' are related by an elementary move, then $R_h[L,J]$ is isotopic to $R_{h'}[L,J]$.*

Proof. Assume there is an elementary move on disc j from h to h'. Extend $H_i(t)\circ f_i \circ H_i(t)^{-1}$ to the unique embedding $\overline{S^3 \setminus \widetilde{L}_i} \to S^3$ with support contained in the image of $H_i(t)$. Let $\xi_i(t)\colon C_u \to S^3$ be its restriction. Define $R_{H(t)}[L,J]$ by the formula:

$$R_{H(t)}[L,J] := \xi_j \circ \left(\xi_1(t)\circ\xi_2(t)\circ\cdots\circ\xi_{j-1}(t)\circ\xi_{j+1}(t)\circ\cdots\circ\xi_{n-1}(t)\circ\xi_n(t)\right).$$

This is well-defined and smooth since $\xi_i(t)(C_u) \subset C_u$ for all $i \neq j$ and $t \in [0,1]$. By design $R_{H(0)}[L,J] = R_h[L,J]$ and $R_{H(1)}[L,J] = R_{h'}[L,J]$.

The construction of the above isotopy is formally analogous to the action of the operad of little 2-cubes on the space of long knots [4].

Given an $(n+1)$-component KGL $L = (L_0, L_1, \ldots, L_n)$ we can represent a disc-system for $\widetilde{L}$ by an embedding $f\colon D \to S^3$ where $D = \bigsqcup_{i=1}^n D_i$ is a disjoint union of 2-discs and $\partial D_i = L_i$. This is because a closed regular neighbourhood of $D \cap C_u$ is a genuine disc-system.

PROPOSITION 11. *Any two disc-systems h and h' for $\widetilde{L}$ are related by a sequence of elementary moves.*

Proof. Assume $f\colon D \to S^3$ is a disc system, intersecting D transversely. Consider the curves of intersection $D \cap f(D) \subset D$. If $D \cap f(D) = \partial D$, f is isotopic to D as a disc-system.

So consider an innermost circle C of $D \cap f(D)$ in D. C is the boundary of two discs $C = \partial S_1$ and $C = \partial S_2$ where $S_1 \subset D$ and $S_2 \subset f(D)$. $S_1 \cup S_2$ is a sphere and so there is a unique 3-ball $B \subset S^3$ with $\partial B = S_1 \cup S_2$ with $B \cap f(D) = S_2$. Let's assume $S_2 = B \cap f(D_i)$ Thus, there is an isotopy of $f(D_i)$, supported in a neighbourhood of B which lowers the number of intersections of $f(D_i)$ with D. This isotopy is a sequence of elementary moves on f, one for every component of $img(f) \cap int(B)$. By induction, we have the necessary collection of elementary moves from f to D.

DEFINITION 12. Let $\mathbf{G}'$ be a sub-graph of $\mathbf{G}_L$, thus it describes some subset of $C_L \mid T$. If M is the union of these submanifolds, and $f\colon M \to S^3$ the untwisted re-embedding, $f(M)$ is the complement of a tubular neighbourhood of some 1-dimensional submanifold $X \subset S^3$. Moreover, the boundary of M are tori either of $C_L \cap M$, or they are edges $e \in \mathbf{G}_L \setminus \mathbf{G}'$ incident to $\mathbf{G}'$. Define $\mathbf{G}_L(\mathbf{G}')$ to be X, with components indexed by $A' \subset A$ corresponding to $C_L \cap M$, and the edges of $\mathbf{G}_L \setminus \mathbf{G}'$ incident to $\mathbf{G}'$. Moreover, since the external peripheral curves of these tori are naturally oriented by the definition of $\mathbf{G}_L$, this gives the components of $\mathbf{G}_L(\mathbf{G}')$ a natural orientation. We call $\mathbf{G}_L(\mathbf{G}')$ the companion link to L for the sub-graph $\mathbf{G}' \subset \mathbf{G}_L$.

PROPOSITION 12. *Given a knot K, let L be the KGL decorating the root of $\mathbf{G}_K$, and let $\mathbf{G}'_1, \ldots, \mathbf{G}'_n$ be the sub-trees of $\mathbf{G}_K$ rooted at the children of L in $\mathbf{G}_K$, then*

$$K \sim (\mathbf{G}_K(\mathbf{G}'_1), \ldots, \mathbf{G}_K(\mathbf{G}'_n)) \bowtie L.$$

Moreover, given a vertex $v \in \mathbf{G}_K$ with $\mathbf{G}'$ the maximal rooted sub-tree of $\mathbf{G}_K$ rooted at v, then $\mathbf{G}_{\mathbf{G}_K(\mathbf{G}')} = \mathbf{G}'$.

We mention, without proof, a result of Eisenbud and Neumann on spliced knots.

THEOREM 1. *Given a knot K let $\Delta_K(t)$ denote the Alexander polynomial of K. Given an integer n let $p(n) = \frac{t^n-1}{t-1}$ for $n \neq 0$ and define $p(0)$ to be 1. If $K = J \bowtie L$ where L is a KGL, then*

$$\Delta_K(t) = \Delta_{L_0}(t) \cdot \prod_{i=1}^{n} \left(p(lk(L_0, L_i))\Delta_{K_i}(t^{lk(L_0,L_i)})\right).$$

Here $lk(L_0, L_i)$ is the linking number between L_0 and L_i where $L = (L_0, L_1, \ldots, L_n)$.

The proof of the above theorem is an application of Theorem 5.3 [10] together with the Torres conditions on the multi-variable Alexander polynomial, which also follows from Theorem 5.3.

A small observation on when splicing produces the unknot.

PROPOSITION 13. *Let $L = (L_0, L_1, \ldots, L_n)$ be a KGL, and $J = (J_1, \ldots, J_n)$ an n-tuple of non-trivial knots. $J \bowtie L$ is the unknot if and only if L is the unlink.*

Proof. '$\Longrightarrow$' By design $\bigsqcup_{i=1}^n C_{J_i}$ naturally embeds in $C_{J \bowtie L}$. As in the proof of Proposition 2, a spanning-disc D for $J \bowtie L$ can be isotoped off $\bigsqcup_{i=1}^n C_{J_i}$, giving a spanning disc D' for L_0 disjoint from $\bigsqcup_{i=1}^n L_i$. If D' intersects the spanning discs for $\bigsqcup_{i=1}^n L_i$ one can modify D' through embedded surgeries along the spanning discs of $\bigsqcup_{i=1}^n L_i$, resulting in a spanning disc for L_0 disjoint from the spanning discs for $\bigsqcup_{i=1}^n L_i$.

'$\Longleftarrow$' Let D be a spanning disc for L_0 disjoint from $\bigsqcup_{i=1}^n L_i$. Then $R_h[L,J](D')$ is a spanning disc for $J \bowtie L$.

PROPOSITION 14. *Assume $L = (L_0, L_1, \ldots, L_n)$ is a non-compound KGL and $J = (J_1, \ldots, J_n)$ is an n-tuple of non-trivial knots. Provided both of the following statements are false:*

- *L is a Hopf link;*
- *L is a key-chain link and at least one of the knots $J_1, \ldots, J_n$ is not prime;*

then the root of $\mathbf{G}_{J \bowtie L}$ is decorated by L, and the maximal sub-trees of $\mathbf{G}_{J \bowtie L}$ rooted at the children of L are $\mathbf{G}_{J_1}, \ldots, \mathbf{G}_{J_n}$ respectively.

Proof. First, consider the case that L is not Seifert fibred. In this case, the complement of $J \bowtie L$ is the union of $R_h[J,L](C_L)$ and C_{J_i} along the tori ∂C_{J_i} for $i \in \{1, 2, \ldots, n\}$. T_i is incompressible in C_{J_i} by the Loop Theorem. It is also incompressible in $R_h[J,L](C_L)$ since if it were not, an unknot would split off L but we have assumed L is non-compound. Thus, the tori T_i are incompressible in $C_{J \bowtie L}$, moreover if we take the union of the collection $\{T_1, \ldots, T_n\}$ together with the JSJ-decompositions of C_{J_i} for $i \in \{1, 2, \ldots, n\}$ we get a collection of tori T such that $C_{J \bowtie L} | T$ consists of Seifert-fibred and atoroidal manifolds. This is a minimal collection by assumption.

So we have reduced to the case C_L Seifert-fibred. Provided the roots of $\mathbf{G}_{J_i}$ are all decorated by non-Seifert fibred spaces, the above argument applies. So assume the root of $\mathbf{G}_{J_i}$ is also Seifert fibred. The fibre-slope of L_i is either p/q if L is $S^{(p,q)}$ or 0 if L is a key-chain link. Consider the possible fibre-slopes of the relevant component of the link decorating the root of $\mathbf{G}_{J_i}$. It could either be ∞ in the case of a key-chain link, or $LCM(r,s)/GCD(r,s)$ in the case of a torus knot or Seifert link. Thus, the only way the Seifert-fibring could extend is the key-chain-key-chain case as in all other cases, the fibre-slopes are not reciprocal (since $p \nmid q$ and $LCM(r,s)/GCD(r,s) \in \mathbf{N}$).

THEOREM 2. *Given a knot K, the companionship tree is a connected splice diagram $\mathbf{G}_K$ with external label $\{*\}$, such that every edge is oriented, and every maximal directed path terminates at v_*, giving $\mathbf{G}_K$ the structure of a rooted tree with root v_*.*

(1) *Each vertex of* $\mathbf{G}_K$ *is labelled by a link from the list:*

(a) *Torus knots* $T^{(p,q)}$ *for* $GCD(p,q) = 1$, $p \nmid q$ *and* $q \nmid p$.

(b) *Seifert links* $S^{(p,q)}$ *for* $GCD(p,q) = 1$, $p \nmid q$.

(c) *Right-handed key-chain links* H^p *for* $p \geq 2$.

(d) *Hyperbolic KGLs.*

(e) *The unknot.*

(2) *Given a vertex* $v \in \mathbf{G}_K$, *then* $\mathbf{G}_K(v)$ *is some KGL indexed by* (A, b) *where the edges of* $\mathbf{G}_K$ *corresponding to* A *are oriented towards* v. *If* v *is not the root, then* b *corresponds to an edge oriented away from* v.

(3) *If any vertex is decorated by a key-chain link* H^p, *none of its children are allowed to be decorated by a key-chain link.*

(4) *A vertex of the tree* $\mathbf{G}_K$ *is allowed to be decorated by the unknot if and only if the tree* $\mathbf{G}_K$ *consists of only one vertex.*

The above properties are complete, in the sense that any graph satisfying the above properties is realisable as $\mathbf{G}_K$ *for some knot* K. *Moreover:*

- *Two knots* K *and* K' *are isotopic if and only if* $\mathbf{G}_K \sim \mathbf{G}_{K'}$.
- *If* $\mathbf{G}_K$ *consists of more than one vertex, then* $K = (J_1, \ldots, J_n) \bowtie L$ *where the root of* $\mathbf{G}_K$ *is labelled by* L *and the knots* $(J_1, \ldots, J_n)$ *correspond to the maximal sub-trees rooted at the children of the root of* $\mathbf{G}_K$.
- *There exists hyperbolic KGLs with arbitrarily many components. Thus one can realise any finite, rooted-tree as* G_K *for some knot* K *in* S^3.

Proof. (1) Proposition 5 lists the Seifert-fibred links and their Brunnian properties. The only Seifert-fibred KGL that we excluded from the list is the Hopf link. All remaining non-compound KGLs are hyperbolic, by Thurston's Hyperbolisation Theorem [32]. (2) follows from the definition. (3) is Proposition 14. (4) The unknot is the only knot whose complement does not have an incompressible boundary. Given a labelled rooted tree satisfying (1)–(4), one constructs the knot K inductively on the height of the tree, using Proposition 14 as the inductive step.

The first two bulleted (•) points follow from Proposition 9, Proposition 2 and Definition 10. The last bulleted point follows from a theorem of Kanenobu [20], as we will explain. Consider a link L indexed by a set A. Then the Brunnian property of L, $\mathcal{U}_L \subset 2^A$ satisfies the 'Brunnian Conditions':

- $S, T \in \mathcal{U}_L$ and $S \cap T = \varnothing \Longrightarrow S \cup T \in \mathcal{U}_L$.
- $\varnothing \notin \mathcal{U}_L$.
- $\{a\} \notin \mathcal{U}_L \ \forall a \in A$.

Debrunner [9] proved the converse, that if $\mathcal{U} \subset 2^A$ is any collection of subsets of a finite set A satisfying the Brunnian Conditions, then there exists L indexed by A such that $\mathcal{U}_L = \mathcal{U}$. Kanenobu went further [20]: there exists L such that $\mathcal{U}_L = \mathcal{U}$ and for all $S \in \mathcal{U}$, L_S is hyperbolic. Moreover, if $A \in \mathcal{U}$, then one can assume all the components of L are unknotted.

If we let $\mathcal{U} = \{A\}$, then Kanenobu's theorem gives us a hyperbolic KGL with $|A|$ components.

Theorem 2 allows one to consider the above class of rooted trees as an index-set for the path-components of the space of embeddings of S^1 in S^3, $\pi_0\mathrm{Emb}(S^1, S^3)$. It is via this indexing that the homotopy type of each component of $\mathrm{Emb}(S^1, S^3)$ is described in the paper [3].

Knots whose complements have non-trivial JSJ-decompositions are quite common. If one generates knots via random walks in $\mathbf{R}^3$ where the direction vector is chosen by a Gaussian distribution, then one typically gets a connect-sum [19].

The observation that if a knot in S^3 has a non-trivial JSJ-decomposition then it is the splice of a link in S^3 with a knot in S^3 was first made in Proposition 2.1 of the Eisenbud-Neumann book [10]. Their point of view on the subject did not keep track of the Brunnian properties of the links involved, in that there is no analogue of Proposition 1 in their work, as their work focuses on links in homology spheres.

COROLLARY 2. *Here are some elementary characterisations of some basic knot operations in terms of splicing.*

- *A knot K is a connected-sum of n non-trivial knots for $n \geq 2$ if and only if $K = J \bowtie H^{n+1}$ where H^{n+1} is the $(n+1)$-component key-chain link, and the knots $J = (J_1, J_2, \ldots, J_n)$ are all non-trivial.*
- *A knot K is a cable knot if and only if K is a splice knot $K = J \bowtie S^{(p,q)}$ for $p \nmid q$ and $GCD(p,q) = 1$, where $S^{(p,q)}$ is the (p,q)-Seifert link.*
- *A knot K is a (untwisted) Whitehead double if $K = J \bowtie L$ where L is the Whitehead link.*

Corollary 2 also appears in Schubert's work [30].

We give various examples of spliced knots and companionship trees. Let F_8 denote the figure-8 knot.

Let W denote the Whitehead link. Let $B = (B_0, B_1, B_2)$ denote the Borromean rings. Let $B(i,j)$ be the 3-component link in S^3 obtained from B

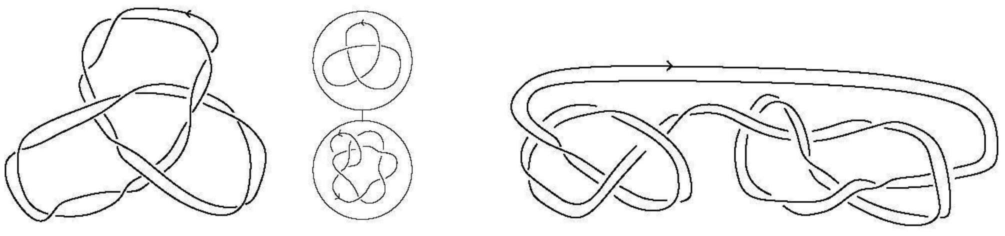

FIGURE 8

by doing i Dehn twists about the spanning disc of B_1 and j Dehn twists about the spanning disc for B_2.

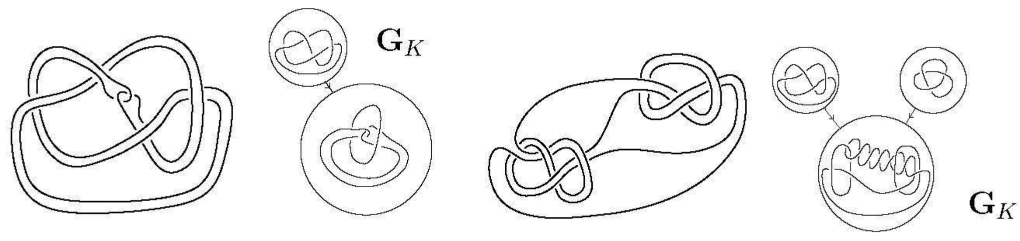

FIGURE 9

The graphs we associate to links in S^3 have more complicated combinatorics for three reasons:

- Link complements are not prime provided the link is split. This results in our graphs being a union of disjoint trees.
- There are link complements with incompressible tori that separate components of the link, thus the associated graphs are not always rooted.
- The tori in the JSJ-decomposition of a link complement are not always knotted.

An example of a link with an unknotted torus in its JSJ-decomposition is given in Figure 10 overleaf.

The rest of this section will be devoted to describing the class of labelled graphs that can be realised as $\mathbf{G}_L$ for a link L in S^3, and how the graphs

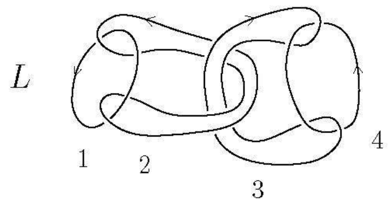

FIGURE 10

behave under splicing. To do this, we identify the local rules that allow us to determine if a link Y can decorate a vertex in a graph $\mathbf{G}_L$ for some L.

Given a vertex $v \in \mathbf{G}_L$ in a companionship graph $\mathbf{G}_L$ of a link L with index set A, we partition the components of $\mathbf{G}_L(v)$ into four classes:

(1) Those which correspond to an oriented edge of $\mathbf{G}_L$ whose terminal point is v.

(2) Those which correspond to an oriented edge of $\mathbf{G}_L$ whose initial point is v.

(3) Those that correspond to an unoriented edge of $\mathbf{G}_L$ incident to v.

(4) The components $\mathbf{G}_L(v)_a$ of $\mathbf{G}_L(v)$ which have labels in the set A.

Components of type (1) through (3) are indexed by a subset of B, the index-set for the tori in the JSJ-decomposition of C_L. Components of $\mathbf{G}_L(v)$ of type (4) are indexed by a subset of A.

PROPOSITION 15 (Local Brunnian Property). *Let L be a link in S^3. Fix a vertex v of $\mathbf{G}_L$ and let $A(v)$ be the index-set for $\mathbf{G}_L(v)$. Let $A_1, A_2, A_3 \subset A(v)$ be the indices corresponding to items* (1), (2) *and* (3) *above. Then the following statements hold:*

(1) $A_1 \in \overline{\mathcal{U}}_{\mathbf{G}_L(v)}$;

(2) $\forall a \in A_2, A_1 \cup \{a\} \notin \overline{\mathcal{U}}_{\mathbf{G}_L(v)}$;

(3) $\forall a \in A_3, A_1 \cup \{a\} \in \overline{\mathcal{U}}_{\mathbf{G}_L(v)}$.

Proof. Let V be the submanifold of $C_L \,|\, T$ corresponding to v. We index the tori of ∂V by the set A'. For each $a \in A_1$, $\partial_a V$ is a torus which bounds a knotted solid torus J_a in S^3 containing V. The collection $\{J_a : a \in A_1\}$ have disjoint complements, so by Proposition 2, $A_1 \in \overline{\mathcal{U}}_{\mathbf{G}_L(v)}$.

Fix $a' \in A_2 \cup A_3$. Then $\partial_{a'} V$ bounds a solid torus in S^3 disjoint from V. This solid torus is a neighbourhood of some knot $K_{a'}$ in S^3. Provided $A_1 = \{a_1, a_2, \ldots, a_n\}$, define $L' = (\mathbf{G}_L(v)_{a'}, \mathbf{G}_L(v)_{a_1}, \ldots, \mathbf{G}_L(v)_{a_n})$. Then by the definition of our identification $V \simeq C_{\mathbf{G}_L(v)}$, $K_{a'} = J \bowtie L'$ where $J = (J_{a_1}, \dots, J_{a_n})$. By Proposition 13, $K_{a'}$ is unknotted if and only if L' is an unlink, if and only if $A_1 \cup \{a'\} \in \overline{\mathcal{U}}_{\mathbf{G}_L(v)}$. This proves points (2) and (3).

A splice diagram will be called valid if it satisfies the Local Brunnian Property. Valid splice diagrams essentially keep track of embedded tori in link complements, as we will show in the next proposition.

Definition 13. Given a link L with index-set A and a family of disjoint embedded tori $T \subset C_L$ indexed by a set B, we define the *splice diagram* associated to the pair (L, T) to be the graph whose underlying vertex-set is $\pi_0(C_L \mid T)$, edge-set is $\pi_0 T \equiv B$, such that each vertex v is decorated by a link following Definition 6 points (3) and (4). We orient the edges of this graph following Definition 5. The notation we use for this graph is $\mathbf{G}_{(L,T)}$.

Notice that if L is not a split link, then $\mathbf{G}_L = \mathbf{G}_{(L,T)}$ provided $T \subset C_L$ is the JSJ-decomposition.

Definition 14. Given links L and L' with index sets A and A' such that $\{a\} \in \overline{\mathcal{U}}_L$ and $a' \in A'$. Let $f \colon S^3 \to S^3$ be an orientation-preserving diffeomorphism such that $f(R[L_a, L'_a](C_{L_a}))$ is a closed tubular neighbourhood of $L'_{a'}$ such that an oriented longitude of $L'_{a'}$ corresponds to an oriented meridian of C_{L_a}. We define the splice of L' and L to be:

$$L' \underset{a'\ a}{\bowtie} L = L'_{A' \setminus \{a'\}} \cup f(L_{A \setminus \{a\}})$$

and we index this link by the set $(A \setminus \{a\}) \sqcup (A' \setminus \{a'\})$. Provided $A \cap A' = \{a\}$ we simply denote the splice by $L' \bowtie L$

Note that if both $\{a\} \in \overline{\mathcal{U}}_L$ and $\{a'\} \in \overline{\mathcal{U}}_{L'}$, then $L' \underset{a'\ a}{\bowtie} L$ and $L \underset{a\ a'}{\bowtie} L'$ are isotopic, so without any harm we can consider the splicing notation for links to be symmetric: $L \underset{a\ a'}{\bowtie} L' = L' \underset{a'\ a}{\bowtie} L$.

Proposition 16. *Given a valid splice diagram* $\mathbf{G}$, *there exist a link* L *and tori* $T \subset C_L$ *such that* $\mathbf{G}_{(L,T)} \sim \mathbf{G}$ *; moreover, the pair* (L, T) *is unique up to isotopy in the sense that if* $\mathbf{G}_{(L',T')} \sim \mathbf{G}$ *then there exists an isotopy* f *from* L *to* L' *such that* $f(T) = T'$.

Proof. The proof of uniqueness is essentially the same as that of Proposition 9. We prove existence by induction. For this we can assume

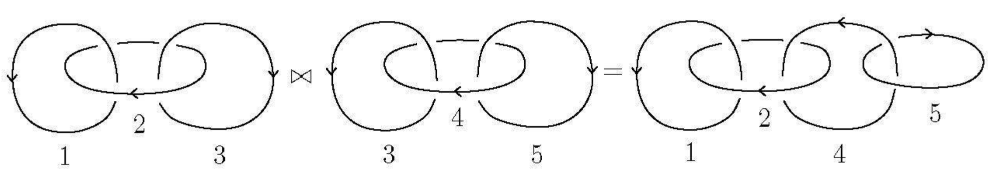

FIGURE 11

that $\mathbf{G}$ is connected, and since the initial step is true by design, we proceed to the inductive step. Let e be an edge of $\mathbf{G}$ with endpoints v_1 and v_2. Then e partitions $\mathbf{G}$ into two sub-graphs $\mathbf{G}'$ and $\mathbf{G}''$.

Provided e is unoriented, then both $\mathbf{G}'$ and $\mathbf{G}''$ are valid splice diagrams and therefore can be realised by pairs (L', T') and (L'', T'') respectively.

If e is oriented, assume its terminal point is v' and initial point is v''. Assume $v' \in \mathbf{G}'$ and $v'' \in \mathbf{G}''$. Then $\mathbf{G}''$ is a valid splice diagram which can be realised by a pair (L'', T''). $\mathbf{G}'$ may not be a valid splice diagram. Consider the sub-graph Y of $\mathbf{G}'$ defined recursively to be the sub-graph of $\mathbf{G}'$ containing v' such that if $w \in Y$ and if f is an oriented edge of $\mathbf{G}'$ whose initial point is in Y, with $A_1 \cup \{f\} \in \overline{\mathcal{U}}_{\mathbf{G}'(w)}$ then the endpoint of f is also in Y. If we unorient all the edges of Y in $\mathbf{G}'$, we obtain a valid splice diagram which can be realised by some pair (L', T').

By the definition of $L' \underset{e\ e}{\bowtie} L''$, T' and T'' naturally correspond to a collection of disjoint tori T in the complement of $L' \underset{e\ e}{\bowtie} L''$. Moreover, $\mathbf{G} = \mathbf{G}_{(L' \underset{e\ e}{\bowtie} L'', T)}$.

DEFINITION 15. Given an oriented edge $e \in \mathbf{G}$ in a valid splice diagram, the sub-graph Y of $\mathbf{G}$ rooted at the terminal-point of e, constructed in the proof of Proposition 16 will be called the *downward consequences* of e.

Given two valid splice diagrams $\mathbf{G}'$ and $\mathbf{G}''$ with external index-sets A', A'' realisable by (L', T') and (L'', T''), provided $a' \in A'$ and $a'' \in A''$, and either $\{a'\} \in \overline{\mathcal{U}}_{L'}$ or $\{a''\} \in \overline{\mathcal{U}}_{L''}$, the splice diagram $\mathbf{G}_{(L' \underset{a'\ a''}{\bowtie} L'', T)}$ constructed in Proposition 16 will be denoted $\mathbf{G}' \underset{a'\ a''}{-\!\!-} \mathbf{G}''$.

We note a convenient global property of companionship graphs, allowing one to determine the Brunnian properties of a link via those of its companions. It also shows how one can determine the edge orientations of $\mathbf{G}_L$ from the vertex labels.

PROPOSITION 17 (Global Brunnian Property). *Given* L *indexed by* A, *any edge* $e \in \mathbf{G}_L$ *separates* $\mathbf{G}_L$ *into two sub-graphs* $\mathbf{G}'$ *and* $\mathbf{G}''$. *Let* $L' = \mathbf{G}_L(\mathbf{G}')$ *and* $L'' = \mathbf{G}_L(\mathbf{G}'')$ *be the companions indexed by the sets* A' *and* A'' *respectively with* $A' \cap A'' = \{e\}$ *and* $(A' \cup A'') \setminus \{e\} = A$. *Then* $\overline{\mathcal{U}}_L = \{B \subset A : (B \cap A' \cup \{e\} \in \overline{\mathcal{U}}_{L'}$ *and* $B \cap A'' \in \overline{\mathcal{U}}_{L''})$ *or* $(B \cap A'' \cup \{e\} \in \overline{\mathcal{U}}_{L''}$ *and* $B \cap A' \in \overline{\mathcal{U}}_{L'})\}$. *If* $v' \in \mathbf{G}'$ *and* $v'' \in \mathbf{G}''$ *are the endpoints of* e, *then* e *is oriented from* v' *to* v'' *(resp.* v'' *to* v'*) if and only if* $\{e\} \notin \overline{\mathcal{U}}_{L'}$ *and* $\{e\} \in \overline{\mathcal{U}}_{L''}$ *(resp.* $\{e\} \notin \overline{\mathcal{U}}_{L''}$ *and* $\{e\} \in \overline{\mathcal{U}}_{L'}$*). Moreover,* e *is unoriented if and only if* $\{e\} \in \overline{\mathcal{U}}_{L'} \cap \overline{\mathcal{U}}_{L''}$. *If* e *is either unoriented or oriented from* v' *to* v'' *(resp.* v'' *to* v'*) then* $\mathbf{G}_{L'} \sim \mathbf{G}'$ *(resp.* $\mathbf{G}_{L''} \sim \mathbf{G}''$*). If* e *is oriented from* v' *to* v'' *(resp.* v'' *to* v'*) then* $\mathbf{G}_{L''}$ *(resp.* $\mathbf{G}_{L'}$*) is equivalent to* $\mathbf{G}''$ *(resp.* $\mathbf{G}'$*) after unorienting the edges of the downward consequences of* e.

Proof. Let $T \subset C_L$, $T' \subset C_{L'}$ and $T'' \subset C_{L''}$ be the tori corresponding to e. We prove the statement about $\overline{\mathcal{U}}_L$, the remaining statements are corollaries of the proof of Proposition 16.

'$\subset$': Given $B \in \overline{\mathcal{U}}_L$, T is compressible in C_{L_B}. By the Loop Theorem, either T' is compressible in $C_{L'_{B\cap A'\cup\{e\}}}$ or T'' is compressible in $C_{L''_{B\cap A''\cup\{e\}}}$.

If T' is compressible in $C_{L'_{B\cap A'\cup\{e\}}}$ then since L_B is an unlink one can choose the spanning discs to be disjoint from T, thus $L'_{B\cap A'}$ and $L''_{B\cap A''}$ are unlinks. L'_a bounds a disc disjoint from $L'_{B\cap A'}$, therefore $L'_{B\cap A'\cup\{e\}}$ is also an unlink. The argument for T'' compressible in $C_{L''_{B\cap A''\cup\{e\}}}$ is formally identical as it is a symmetric argument.

'$\supset$': Let $\{M', M''\} = C_L \,|\, T$ with $f' : M' \to C_{L'}$ and $f'' : M'' \to C_{L''}$ the untwisted re-embeddings. If $B \cap A' \cup \{e\} \in \overline{\mathcal{U}}_{L'}$ and $B \cap A'' \in \overline{\mathcal{U}}_{L''}$, then if D' and D'' are the spanning discs for $L'_{B\cap A'}$ and $L''_{B\cap A''}$, $f'^{-1}(D') \cup f''^{-1}(D'')$ are spanning discs for L_B. This is also a symmetric argument.

Notice that in our proof of the Global Brunnian Property, we did not use the incompressibility of the tori T, thus it applies equally well to valid splice diagrams.

EXAMPLE 3. Consider the link $L = (L_1, L_2)$ in Figure 12.

Let $\mathbf{G}'$ be the sub-graph of $\mathbf{G}_L$ obtained by deleting the vertex labelled by the (left-handed) trefoil indexed by c. Compare with Figure 13.

There is another local property satisfied by companionship graphs. Only certain Seifert-fibred links may be adjacent in $\mathbf{G}_L$. As we have seen, given an edge $e \in \mathbf{G}_L$ the fibre-slopes of the components of the two adjacent

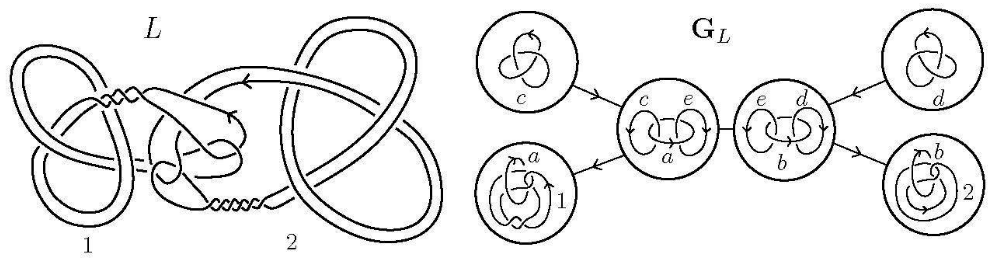

FIGURE 12

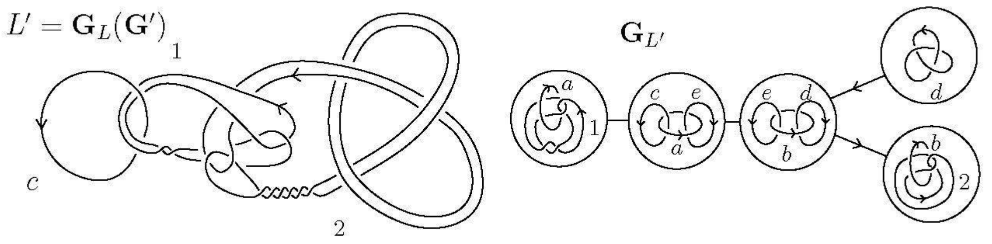

FIGURE 13

links corresponding to e are never multiplicative inverses of each other (see Propositions 7 and 14).

LEMMA 2 (Fibre-slope Exclusion Property). *A vertex decorated by an unknot or a Hopf link can not be adjacent to any other vertex in a companionship graph* $\mathbf{G}_L$ *for any link* L.

Given two adjacent vertices in a companionship graph $\mathbf{G}_L$ *decorated by Seifert-fibred links whose unoriented isotopy class representatives are* $S(p,q \mid X)$ *and* $S(a,b \mid Z)$ *respectively, there are several combinations that can not occur for any link* L*, which we list.*

1. $S(p,q \mid \{*_1\})$ *and* $S(a,b \mid X)$ *with edge corresponding to* $*_1 \in S(p,q \mid \{*_1\})$ *and a regular fibre of* $S(a,b \mid X)$*, provided* $\frac{q}{p} = \frac{LCM(a,b)}{GCD(a,b)}$.
2. $S(p,q \mid X \cup \{*_1\})$ *and* $S(a,b \mid Z \cup \{*_2\})$ *with edge corresponding to* $*_1$ *and* $*_2$ *respectively, such that* $\frac{p}{q} = \frac{a}{b}$.
3. H^p *and* H^q *for* $p,q \geq 2$ *with edge corresponding to components of* H^p *and* H^q *having different fibre-slopes, i.e. a 'key' and a 'keyring' component respectively.*

The Hopf links and unknots are isolated since one can obtain any rational number (or ∞) as the fibre-slope of some Seifert fibring of the complement. All other complements have unique Seifert-fibrings (Proposition 1).

Lemma 2 can be recast into a splicing rule.

DEFINITION 16. Given links of the form $S(p,q \mid X)$ and $S(a,b \mid Z)$, let r denote any regular fibre of $S(p,q \mid X)$ or $S(a,b \mid Z)$. Let L be a link with index-set A, and let O be the unknot with index set $\{*\}$. The following is the list of 'exceptional splices':

1. $S(a,b \mid X) \underset{r\ *_1}{\bowtie} S(p,q \mid \{*_1\}) = S(\frac{GCD(a,b)+GCD(p,q)-1}{GCD(a,b)}(a,b) \mid X)$ provided $\frac{q}{p} = \frac{LCM(a,b)}{GCD(a,b)}$.
2. $S(p,q \mid X \cup \{*_1\}) \underset{*_1\ *_2}{\bowtie} S(a,b \mid Z \cup \{*_2\}) = S(\frac{GCD(a,b)+GCD(p,q)}{GCD(a,b)}(a,b) \mid (X \setminus \{*_1\}) \cup (Z \setminus \{*_2\}))$ provided $\frac{p}{q} = \frac{a}{b}$.
3. $H^p \underset{k\ g}{\bowtie} H^q = H^{p+q-1}$ provided k and g represent a 'key' and a 'keyring' of H^p and H^q respectively.
4. $H^1 \bowtie L = L$ for any choice of components, provided H^1 is the right-handed Hopf link. If H^1 is left-handed, then $H^1 \bowtie L$ consists is L with one component's orientation reversed.
5. $L \underset{a\ *}{\bowtie} O = L_{A\setminus\{a\}}$ for all $a \in A$.

PROPOSITION 18. *Given a valid splice diagram* $\mathbf{G}$ *satisfying the fibre-slope exclusion property such that each vertex* $v \in \mathbf{G}$ *is labelled by either a Seifert-fibred or hyperbolic link* $\mathbf{G}(v)$, *there exists a unique link* L *(up to isotopy) such that* $\mathbf{G}_L \sim \mathbf{G}$. *Moreover, given any two links* L' *and* L'' *with* $a' \in A'$ *and* $a'' \in A''$ *satisfying either* $\{a'\} \in \overline{\mathcal{U}}_{L'}$ *or* $\{a''\} \in \overline{\mathcal{U}}_{L''}$, $\mathbf{G}_{L' \underset{a'\ a''}{\bowtie} L''} = \mathbf{G}_{L'} \underset{a'\ a''}{——} \mathbf{G}_{L''}$ *provided* $\mathbf{G}_{L'} \underset{a'\ a''}{——} \mathbf{G}_{L''}$ *satisfies the Fibre-Slope Exclusion Property.*

Proof. This follows immediately from Proposition 16 and the proof of Proposition 14.

We proceed to investigate the companionship tree of $L' \underset{a'\ a''}{\bowtie} L''$ for arbitrary links L' and L''. To describe the case where one of L' or L'' is the unknot, we will need the operations of 'splitting' and 'deletion' for valid splice diagrams, defined below.

DEFINITION 17. Consider a valid splice diagram $\mathbf{G}$ with external index-set A, vertex-set V and edge-set E.

If $v \in V$ is such that $\mathbf{G}(v)$ is split, we will define a valid splice-diagram called the *splitting* of $\mathbf{G}$ at v, denoted $\mathbf{G} \mid v$. Let X be the index-set for $\mathbf{G}(v)$, and let $X = \bigsqcup_{i=1}^k X_i$ be the partition of X so that $C_{\mathbf{G}(v)} \simeq C_{\mathbf{G}(v)_{X_1}} \# \cdots \# C_{\mathbf{G}(v)_{X_k}}$ is the prime decomposition of $C_{\mathbf{G}(v)}$. Define $\mathbf{G} \mid v$ to be the splice diagram whose vertex-set is $(V \setminus \{v\}) \sqcup \left(\bigsqcup_{i=1}^k \pi_0 C_{\mathbf{G}(v)_{X_i}}\right)$, and whose edge set is E. We label the vertices $w \in V \setminus \{v\}$ by $\mathbf{G}(w)$, and the vertices $\pi_0 C_{\mathbf{G}(v)_{X_i}}$ by $\mathbf{G}(v)_{X_i}$.

If $a \in A$, we define a valid splice-diagram called the *deletion* of a in $\mathbf{G}$, denoted $\mathbf{G}.a$. Let $\mathbf{G}'$ be the maximal sub-graph of $\mathbf{G}$ with vertex-set $V \setminus v_a$. If A' is the index-set of $\mathbf{G}(v_a)$, let $\mathbf{G}'' = \mathbf{G}_{L_{A' \setminus \{a\}}}$. Let E' denote the edges of $\mathbf{G}$ that are not edges of $\mathbf{G}'$, then $\mathbf{G}' \sqcup \mathbf{G}''$ is naturally a valid splice-diagram once we append the edges E'.

If one thinks in terms of the pair (L, T) such that $\mathbf{G}_{(L,T)} = \mathbf{G}$, splitting corresponds to finding a 2-sphere in the complement of $L \cup T$ that separates components, while $\mathbf{G}.a$ corresponds to deletion of the component L_a and adding appending the tori from the JSJ-decomposition of $L_{A' \setminus \{a\}}$.

PROPOSITION 19. *Given two links L' and L'' with index sets A' and A'' respectively such that $L' \underset{a'\ a''}{\bowtie} L''$ is defined, and provided $\mathbf{G}_{L'}(v_{a'}) \underset{a'\ a''}{\bowtie} \mathbf{G}_{L''}(v_{a''})$ is not an exceptional splice, $\mathbf{G}_{L' \underset{a'\ a''}{\bowtie} L''} = \mathbf{G}_{L'} \underset{a'\ a''}{--} \mathbf{G}_{L''}$.*

In the case of an exceptional splice 1. *through* 4., *one obtains $\mathbf{G}_{L' \underset{a'\ a''}{\bowtie} L''}$ from $\mathbf{G}_{L'} \underset{a'\ a''}{--} \mathbf{G}_{L''}$ by replacing the sub-graphs (see Definition 16):*

(1) $S(a,b \mid X)\ -\ S(p,q \mid \{*_1\})$ *by* $S\left(\frac{GCD(a,b)+GCD(p,q)-1}{GCD(a,b)}(a,b) \mid X\right)$.

(2) $S(p,q \mid X \cup \{*_1\})\ -\ S(a,b \mid Z \cup \{*_2\})$ *by* $S\left(\frac{GCD(a,b)+GCD(p,q)}{GCD(a,b)}(a,b) \mid (X \setminus \{*_1\}) \cup (Z \setminus \{*_2\})\right)$.

(3) $H^p\ -\ H^q$ *by* H^{p+q-1}.

(4) $H^1\ -\ L$ *by* L*, or* L *with the corresponding orientation reversed if* H^1 *is the negatively oriented Hopf link.*

We call the operations (1) *through* (4), *together with* (5) *below, 'elementary reductions'.*

In the case (5) *of an exceptional splice with an unknot $L'\ -\ O$, then $\mathbf{G}_{L'\ -\ O}$ is obtained from $\mathbf{G}_{L'}.a$ by performing (recursively) all possible elementary reductions and splittings until the there are no more available to do (compare with Figures 12 and 13).*

Proof. All cases except case (5) are direct consequences of Lemma 2. In case (5), $\mathbf{G}_{L'}.a'$ describes a collection of tori T dividing the complement of $L'_{A'\setminus\{a'\}}$ into atoroidal and Seifert-fibred spaces. $C_{L'_{A'\setminus\{a'\}}}$ may not be prime, and the tori may be compressible. If S be a sphere in $C_{L'_{A'\setminus\{a'\}}}$ intersecting T transversely separating components of $L'_{A'\setminus\{a'\}}$, and D is an innermost disc, then this disc is a spanning disc for a component of one of the links decorating $\mathbf{G}_{L'}.a'$, thus we can perform a reduction of type (5). If there is no innermost disc, we can perform a splitting. So after performing enough type (5) moves and splittings, we are reduced to a disjoint union of diagrams that describe incompressible tori in irreducible link complements that split the link complement into Seifert-fibred and atoroidal manifolds. Any minimal such collection is the JSJ-decomposition, and moves (1) through (4) are by design what it takes to get to that minimal collection.

Provided one never needs to use step (5), Proposition 19 gives a rather simple description of the companionship graph of a splice. A good example of Proposition 19 is obtained by deleting a component of the link in L in Example 3.

In the case of a splice with an unknot $L' - O$ where L' is hyperbolic, Proposition 19 says nothing. We mention a result of Thurston that gives some insight into this situation. First, an example.

EXAMPLE 4. The link L' in Figure 14 contains the link L from Example 3 as a sublink. L' is hyperbolic with hyperbolic volume approximately 42.7594. The link L from Example 3 has two hyperbolic companion links, both with hyperbolic volume approximately 3.663. This is not a coincidence, as we will explain.

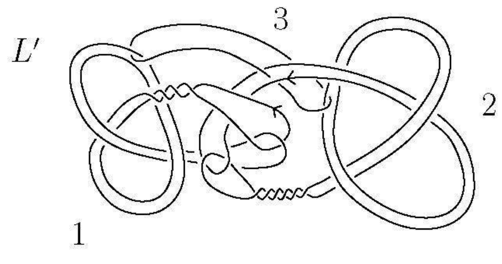

FIGURE 14

DEFINITION 18. Given a link L, the *Gromov Norm* of L is defined to be the sum

$$|L| = \sum_{v \in \mathbf{G}_L} \begin{cases} Vol(S^3 \setminus \mathbf{G}_L(v)) & \text{if } S^3 \setminus \mathbf{G}_L(v) \text{ is hyperbolic,} \\ 0 & \text{if } S^3 \setminus \mathbf{G}_L(v) \text{ is Seifert-fibred.} \end{cases}$$

where $Vol(\cdot)$ is the hyperbolic volume.

See [12] for a proper definition of Gromov Norm. That the proper definition reduces to the one above in the case of link complements is a theorem of Thurston [34]. Thurston also proved that if a link L' is a proper sub-link of a hyperbolic link L then $|L'| < |L|$. Thus, if L' is obtained from an arbitrary link by deleting a component, $|L'| \leq |L|$ and one has equality if and only if when constructing $\mathbf{G}_{L'}$ using the procedure in Proposition 19, one must never delete a component of a hyperbolic companion to L.

We end with one note on the computability of the companionship graph $\mathbf{G}_L$. Typically one starts with a diagram for L and then constructs a (topological) ideal triangulation of its complement, with an algorithm such as the one implemented in SNAPPEA [37] or ORB [15]. The algorithm of Jaco, Letscher, Rubinstein [17] computes the JSJ-graph G_L from the ideal triangulation. Link complements, and more generally Haken manifolds are known to have solvable word problems [36]. These Haken-manifold techniques extend to an isotopy classification of knots and links that seems (at present) to be not practical to implement. Commonly-used computer programs called SNAPPEA [37] and ORB [15] frequently find the hyperbolic structures on hyperbolisable 3-manifolds (and orbifolds). Once SNAPPEA has the hyperbolic metric, it then finds the canonical polyhedral decomposition [11] from which it can determine if two hyperbolic links are isotopic by a simple combinatorial check of their polyhedral decompositions. So in practise one can frequently use SNAPPEA to determine whether any two links are isotopic. It's unfortunate that there is as of yet no formal justification for the effectiveness of programs such as SNAPPEA and ORB. The Manning Algorithm [24] takes a hyperbolisable 3-manifold with a solution to its word problem and produces a hyperbolic metric, but this requires the usage of solutions to word problems for Haken manifolds, which typically have very long run-times. For recognition of connect-sums, the 3-sphere recognition algorithm of Rubinstein has been beautifully implemented in the computer software REGINA, by Ben Burton [7].

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(Reçu le 9 mai 2006)

Ryan Budney

IHES Le Bois-Marie
35, route de Chartres
F-91440 Bures-sur-Yvette
France

Mathematics and Statistics
University of Victoria
PO Box 3045 STN CSC
Victoria, B.C.
Canada V8W 3P4
e-mail: budney@ihes.fr

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Page vide