Thus the second smallest lattice is given by the maximal order with $D = -4$ (the square lattice) and the third and fourth smallest lattices by $D = -7$ and $D = -15$ respectively.

REMARK 3. The inequality (5.5) is quite subtle. Let $N_k = 2 \cdot 3 \cdots p_k$ be the product of the first k primes, then if the Riemann Hypothesis is true (5.5) is false for every integer n with $n = N_k$. On the other hand, if the Riemann Hypothesis is false then there are infinitely many integers k for which $n = N_k$ does satisfy (5.5). See Nicolas [23] for a proof of this interesting result.

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