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THE FEUERBACH CIRCLE AND ORTHOCENTRICITY IN NORMED PLANES

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ABSTRACT. E. Asplund and B. Grünbaum [2] derived various results related to the three-circles theorem and the corresponding type of orthocentricity (called $\mathcal{C}$-orthocentricity) in strictly convex, smooth Minkowski planes. Continuing their investigations, we present new theorems on $\mathcal{C}$-orthocentricity, on the Feuerbach circle and other circles related to Minkowskian triangles and on analogous configurations corresponding to quadrangles and n-gons instead of triangles. In contrast to the considerations in [2], smoothness of the considered norm is not necessarily required. The relation between $\mathcal{C}$-orthocentricity and isosceles (or James) orthogonality is also clarified, and finally some results on quadrangles in arbitrary normed planes are obtained.

1. INTRODUCTION

Let $\mathbf{A}_2$ be the (real) affine plane, and let $\mathcal{C}$ denote a simple, closed and convex curve in $\mathbf{A}_2$ which is centred at the origin O. The curve $\mathcal{C}$ induces a *norm* $\|\cdot\|$ whose *unit circle* is $\mathcal{C}$ and whose *unit disc* is the convex hull of $\mathcal{C}$, see, e.g., [13, Section 2], [16, Ch. 4], and [12, Section 2.2]. The pair $(\mathbf{A}_2, \mathcal{C})$ can thus be considered as a two-dimensional real normed linear space, also called a (*normed* or) *Minkowski plane*. A Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, for which $\mathcal{C}$ does not contain a non-degenerate line segment, is called *strictly convex*. Basic references on the geometry of Minkowski planes and n-dimensional Minkowski spaces (i.e., n-dimensional real normed linear spaces) are Thompson's book [16] and the papers [13], [11], and [12].

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As observed in [16], not too many results are known in the spirit of "elementary geometry of Minkowski planes". Our paper refers to this topic.

Any curve C in $\mathbf{A}_2$ of type $x + \lambda\, \mathcal{C}$, where $x \in \mathbf{A}_2$ and λ is a positive real number, is called a *(Minkowski) circle* with *center* x and *radius* λ, and we will denote it by $C(x, \lambda)$. In the sequel, we denote the *line* through $x \in \mathbf{A}_2$ and $y \in \mathbf{A}_2$, $x \neq y$, by $\langle xy \rangle$, and the *closed segment* from x to y by $[xy]$.

The *three-circles theorem* or Ţiţeica's theorem (cf. the survey [9]), sometimes also called Johnson's theorem, says that if three congruent circles in the Euclidean plane pass through a common point, then their other three pairwise intersection points lie on a fourth circle of the same radius. Furthermore, the four mentioned points form an orthocentric point system (i.e., each is the intersection point of the three altitudes of the triangle formed by the other three), and the same holds for the four circle centers. These and further properties as well as applications of the described figure can be found in the books [8, § 104], [6, § 2.1], [15, Ch. 10], and [19, pp. 50–51], and this arrangement of four congruent circles plays an important role in elementary geometry (orthocenters, inversive geometry, applications of complex numbers), descriptive geometry (spatial interpretations of planar theorems, Pohlke's theorem), configurations (Clifford's chain of theorems; see [10]), combinatorial geometry (Erdős-type questions), and convexity (equilateral zonotopes); see again the survey [9] for all these aspects.

E. Asplund and B. Grünbaum [2] extended the three-circles theorem to strictly convex, smooth Minkowski planes and obtained also related results on Minkowskian analogues of the nine-point circle (Feuerbach circle) of given triangles, which occurs in such normed planes only as a six-point circle.

We will extend the results from [2] to strictly convex normed planes which are not necessarily smooth (see also our figures), and we will complete the results of Asplund and Grünbaum by giving many conceptually new theorems. In addition, we will clarify the relation of the so-called $\mathcal{C}$-orthocentricity (as defined by Asplund and Grünbaum with the help of the three-circles theorem) and the notion of *James orthogonality* (see [7]) for normed planes. Analogous statements for Minkowskian n-gons are also proved, and finally results on the eight-point circle of quadrangles in Minkowski planes are obtained.

Most of our results cannot be extended to arbitrary Minkowski planes (with non-strict convexity of the unit disc), since then our basic Lemmas 2.1 and 2.2 below would not hold. However, at least in the final part we prove some statements referring to *all* normed planes.

2. $\mathcal{C}$-ORTHOCENTRIC SYSTEMS OF POINTS IN STRICTLY CONVEX MINKOWSKI PLANES

It is well known that the following results hold in any strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, see [2], [1, §3, p. 32], and [16, §4.1, p. 104]:

LEMMA 2.1. *Given any three non-collinear points, there exists at most one circle containing them.*

LEMMA 2.2. *If $x_1 \neq x_2$ and $y_1, y_2 \in C(x_1, \lambda) \cap C(x_2, \lambda)$ with $y_1 \neq y_2$, then $x_1 + x_2 = y_1 + y_2$.*

THEOREM 2.3 (Three-circles theorem). *Let p_1, p_2, p_3 be distinct points of $C(x, \lambda)$, and let $C(x_i, \lambda)$, $i = 1, 2, 3$, be three circles different from $C(x, \lambda)$ each of which contains two of the three points p_i. Then $\bigcap_{i=1}^{3} C(x_i, \lambda)$ is not empty and consists of precisely one point p; see Figure 1.*

REMARK 2.4. The point p is called the $\mathcal{C}$-*orthocenter*[1]) of the triangle $p_1p_2p_3$. Moreover,

$$p = p_1 + p_2 + p_3 - 2x. \tag{1}$$

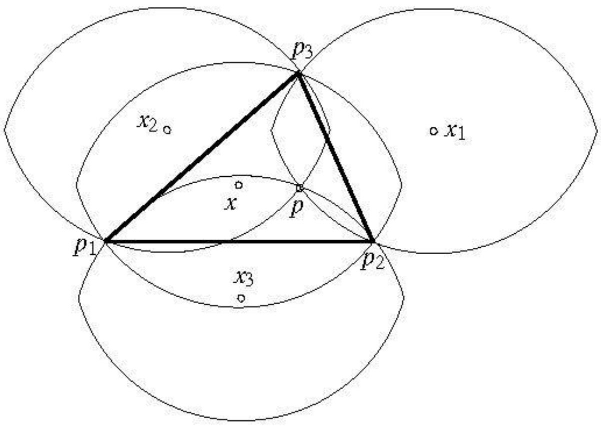

FIGURE 1

[1]) For another concept of triangle orthocenters in Minkowski planes see [17].

REMARK 2.5. Not every triangle in a strictly convex normed plane has a circumcircle (see Figure 28 in [13]) and a $\mathcal{C}$-orthocenter, but if such a plane is also smooth, then the existence of a $\mathcal{C}$-orthocenter is guaranteed; see again [2].

REMARK 2.6. In the Euclidean case, the $\mathcal{C}$-orthocenter of any triangle is the common point of the altitudes of this triangle, i.e., the classical orthocenter of this triangle.

REMARK 2.7. A set of four points, one of which is the $\mathcal{C}$-orthocenter of the other three, is called a *$\mathcal{C}$-orthocentric system*. Each point of such a system is the $\mathcal{C}$-orthocenter of the triangle of the other three; cf. Figure 1. The circumradii of the four triangles formed by the points of a $\mathcal{C}$-orthocentric system are the same.

It is known that in the Euclidean plane the circumcenters of the four triangles obtained by the points of any orthocentric system form another orthocentric system congruent to the first (see [18, p. 165], [9], and Figure 2). Furthermore, the centroids of any three points of an orthocentric system also form an orthocentric system, similar to the first and having a third of its size; see again [18, p. 165].

We will prove that these two properties of orthocentric systems remain true in all strictly convex Minkowski planes. We first prove the following

LEMMA 2.8. *Let $\{p_1, p_2, p_3, p_4\}$ be a $\mathcal{C}$-orthocentric system in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$. Let x_i be the circumcenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $p_k + x_k = p_l + x_l$; see Figure 2.*

Proof. From Remark 2.4 we have

$$p_l = p_i + p_j + p_k - 2x_l .$$

Using

$$p_i + p_j = x_k + x_l ,$$

see Lemma 2.2, we get

$$p_l + 2x_l = p_i + p_j + p_k \Longleftrightarrow p_l + 2x_l = x_k + x_l + p_k \Longleftrightarrow p_l + x_l = x_k + p_k . \qquad \square$$

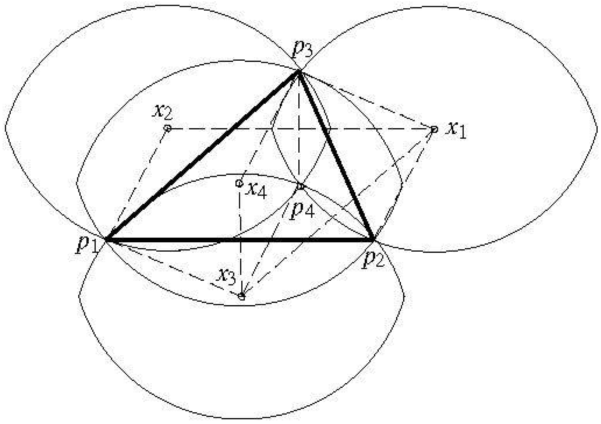

FIGURE 2

THEOREM 2.9. *Let* $\{p_1, p_2, p_3, p_4\}$ *be a* $\mathcal{C}$*-orthocentric system in a strictly convex Minkowski plane* $(\mathbf{A}_2, \mathcal{C})$. *If* x_i *is the circumcenter of the triangle* $p_j p_k p_l$, *where* $\{i, j, k, l\} = \{1, 2, 3, 4\}$, *then* $\{x_1, x_2, x_3, x_4\}$ *is also a* $\mathcal{C}$*-orthocentric system and*

$$p_i - p_j = x_j - x_i \,. \tag{2}$$

Proof. Applying again Remark 2.4, we get

$$p_i = p_j + p_k + p_l - 2x_i \quad \text{and} \quad p_j = p_k + p_l + p_i - 2x_j \,.$$

From this relation (2) follows immediately. Moreover, the $\mathcal{C}$-orthocentricity of $\{p_1, p_2, p_3, p_4\}$ implies that

$$p_i = C(x_j, \lambda) \cap C(x_k, \lambda) \cap C(x_l, \lambda) \,,$$

where $\lambda > 0$. Thus, for $s = j, k, l$ we have

$$\|p_i - x_s\| = \lambda \,. \tag{3}$$

This means that $x_j, x_k, x_l \in C(p_i, \lambda)$, which completes the proof. □

It is worth mentioning that the point set $\{x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4\}$, as in the Euclidean situation, forms the vertex set of the image of a 3-cube under parallel projection onto $\mathbf{A}_2$, yielding a useful "spatial interpretation" of this planar configuration; see Figure 2 and [9].

The next corollaries follow from equation (3).

COROLLARY 2.10. *Let $\{p_1, p_2, p_3, p_4\}$ be a $\mathcal{C}$-orthocentric system in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, and x_i be the circumcenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The circumradius of the triangles formed by the points of the system $\{p_1, p_2, p_3, p_4\}$ equals the circumradius of the triangles formed by the points of the system $\{x_1, x_2, x_3, x_4\}$.*

COROLLARY 2.11. *Let $\{p_1, p_2, p_3, p_4\}$ be a $\mathcal{C}$-orthocentric system in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, and x_i be the circumcenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then p_i is the circumcenter of the triangle $x_j x_k x_l$.*

The following proposition gives one more relationship between the $\mathcal{C}$-orthocentric systems $\{p_1, p_2, p_3, p_4\}$ and $\{x_1, x_2, x_3, x_4\}$.

PROPOSITION 2.12. *Let $\{p_1, p_2, p_3, p_4\}$ be a $\mathcal{C}$-orthocentric system in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, and x_i be the circumcenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then the midpoints of the segments $[p_i x_i]$ coincide.*

Proof. Let $f_i = \frac{1}{2}(p_i + x_i)$. Then Lemma 2.8 yields $f_i = f_j$. □

The next theorem is related to the system of centroids of the triangles formed by the points of an orthocentric system.

THEOREM 2.13. *Let $\{p_1, p_2, p_3, p_4\}$ be a $\mathcal{C}$-orthocentric system in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$. Let m_i be the centroid of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $\{m_1, m_2, m_3, m_4\}$ is also a $\mathcal{C}$-orthocentric system, and we have*

$$m_i - m_j = \frac{1}{3}(p_j - p_i), \tag{4}$$

see Figure 3.

Proof. The relation (4) is obvious. Now let $C(x_i, \lambda)$ be the circumcircle of the triangle $p_j p_k p_l$, and $y_i = \frac{2}{3} p_i + \frac{1}{3} x_i$. We will prove that the circle $C\left(y_i, \frac{1}{3}\lambda\right)$ contains the point m_j. Indeed,

$$\begin{aligned}\|y_i - m_j\| &= \left\| \frac{2}{3} p_i + \frac{1}{3} x_i - \frac{1}{3}(p_k + p_l + p_i) \right\| = \frac{1}{3} \|p_i - p_k - p_l + x_i\| \\ &= \frac{1}{3} \|(p_j + p_k + p_l - 2x_i) - p_k - p_l + x_i\| = \frac{1}{3} \|p_j - x_i\| = \frac{1}{3}\lambda,\end{aligned}$$

by Remark 2.4. Analogously we have $m_k, m_l \in C\big(y_i, \frac{1}{3}\lambda\big)$, and the theorem is proved. □

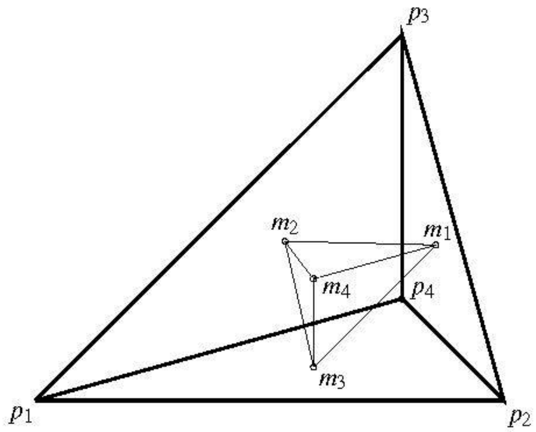

FIGURE 3

The last theorem in this section refers to quadruples of concyclic points and the C-orthocenters of the four triangles formed by these quadruples.

THEOREM 2.14. *Let p_1, p_2, p_3, p_4 be four distinct points in a strictly convex Minkowski plane, lying on the circle $C(x, \lambda)$. Let h_i be the C-orthocenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then*

(i) *the points h_i, h_j, h_k, p_l form a C-orthocentric system;*

(ii) *the points h_i, h_j, p_k, p_l lie on a circle with radius λ.*

Proof. (i) It is sufficient to prove that p_l is the C-orthocenter of the triangle $h_i h_j h_k$. Without loss of generality we may assume that $x = O$. Then, by Remark 2.4,

$$
\begin{aligned}
h_i &= p_j + p_k + p_l = p_1 + p_2 + p_3 + p_4 - p_i\,,\\
h_j &= p_k + p_l + p_i = p_1 + p_2 + p_3 + p_4 - p_j\,,\\
h_k &= p_l + p_i + p_j = p_1 + p_2 + p_3 + p_4 - p_k\,.
\end{aligned} \tag{5}
$$

We will prove that $y = p_1 + p_2 + p_3 + p_4$ is the circumcenter of $h_i h_j h_k$. Indeed, in view of (5) we have

$$\|y - h_s\| = \|(p_1 + p_2 + p_3 + p_4) - (p_1 + p_2 + p_3 + p_4 - p_s)\| = \|p_s\| = \lambda, \tag{6}$$

where $s = i, j, k$. Therefore the C-orthocenter p^* of $h_i h_j h_k$ is

$$p^* = h_i+h_j+h_k-2y = 3(p_1+p_2+p_3+p_4)-(p_i+p_j+p_k)-2(p_1+p_2+p_3+p_4) = p_l\,,$$

see Remark 2.4 and (5).

(ii) It is easy to check that $h_i, h_j, p_k, p_l \in C(p_k + p_l, \lambda)$. Thus the desired statement is proved. □

The next corollary follows directly from equation (6).

COROLLARY 2.15. *Let p_1, p_2, p_3, p_4 be four pairwise-distinct points in a strictly convex Minkowski plane, lying on the circle $C(x, \lambda)$, and h_i be the $\mathcal{C}$-orthocenter of the triangle $p_jp_kp_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then the points h_1, h_2, h_3, h_4 lie on a circle with the same radius λ.*

REMARK 2.16. Under the assumption in the proof of Theorem 2.14 (i.e., $x \equiv O$), the center of the circle determined by h_1, h_2, h_3, h_4 is

$$y = p_1 + p_2 + p_3 + p_4\,.$$

REMARK 2.17. The circle determined by h_1, h_2, h_3, h_4 (see Theorem 2.14) can be derived in another way. In [2] it is proved that if p_1, p_2, p_3, p_4 lie on the circle $C(O, \lambda)$ and four circles of the same size are drawn through p_1 and p_2, p_2 and p_3, p_3 and p_4, as well as p_4 and p_1, then the other four intersection points of these new circles lie on the circle $C(p_1+p_2+p_3+p_4, \lambda)$. For Euclidean circles of arbitrary size this statement is known as Miquel's theorem; see, e.g., [18, p. 151].

3. Relations between $\mathcal{C}$-orthocenters and James orthogonality

As mentioned in the previous section, in the Euclidean case the $\mathcal{C}$-orthocenter of any triangle is the common point of the altitudes (i.e., the classical orthocenter) of this triangle. Thus a natural question arises: is the line through an apex and the $\mathcal{C}$-orthocenter of any triangle in a Minkowski plane orthogonal[2]) to the opposite side? The present section is devoted to this problem.

The vector $x \in (\mathbf{A}_2, \mathcal{C})$ is *James orthogonal* (or *isosceles orthogonal*) to $y \in (\mathbf{A}_2, \mathcal{C})$ if

[2]) It is well known that in Minkowski planes there are different concepts of orthogonality. For a variety of such concepts see, e.g., [1, §3–4, and §7–8], [3], and [16, §3.5].

$$\|x+y\| = \|x-y\|\,,$$

see [7]. If x is James orthogonal to y, we will write $x \# y$. It is obvious that this relation has the following properties:

$$x \# y \iff y \# x \iff \alpha x \# \alpha y \quad \text{for any } \alpha \in \mathbf{R} \iff x \# -y\,,$$

see [1, p. 24].

LEMMA 3.1. *If $x, y \in \mathbf{A}_2$, then*

$$\|x\| = \|y\| \iff x+y \# x-y\,.$$

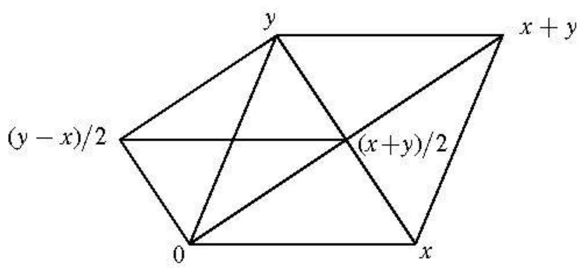

FIGURE 4

Proof. It is evident (see Figure 4) that

$$x+y \# x-y \iff \frac{1}{2}(x+y) \# \frac{1}{2}(x-y) \iff \|x\| = \|y\|\,. \qquad \square$$

THEOREM 3.2. *Let $\{p_1, p_2, p_3, p_4\}$ be a $\mathcal{C}$-orthocentric system in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$. Then*

$$p_i - p_j \# p_k - p_l\,,$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. Let x_i be the circumcenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Lemma 2.2 implies that $p_i x_l p_j x_k$ is a parallelogram. Since x_l is the circumcenter of $p_i p_j p_k$, we have $\|x_l - p_i\| = \|x_l - p_j\|$. Thus, by Lemma 3.1,

$$x_l - x_k \# p_i - p_j\,.$$

On the other hand, by Lemma 2.8 we have $x_l - x_k = p_k - p_l$. Therefore $p_k - p_l \# p_i - p_j$. $\square$

Reformulating the above theorem we get a statement that directly shows the relationship between $\mathcal{C}$-orthogonality in the sense of Asplund and Grünbaum and James orthogonality.

THEOREM 3.3. *For any triangle with $\mathcal{C}$-orthocenter in a strictly convex Minkowski plane the following holds: a vector, determined by one of its vertices and the $\mathcal{C}$-orthocenter, is James orthogonal to the opposite triangle side.*

4. THE FEUERBACH CIRCLE IN STRICTLY CONVEX MINKOWSKI PLANES

It is well known that in the Euclidean plane the feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices to the orthocenter lie on the same circle, called the *nine-point circle* [3]) of the triangle; see Figure 5. (For more details about the nine-point circle we refer to [5, §1.8, p. 20] and [18, p. 159].) This statement can be partially generalized to all strictly convex Minkowski planes. The idea of this generalization is also due to Asplund and Grünbaum, and the next theorem as well as the three remarks after it give a brief review of the results obtained in [2].

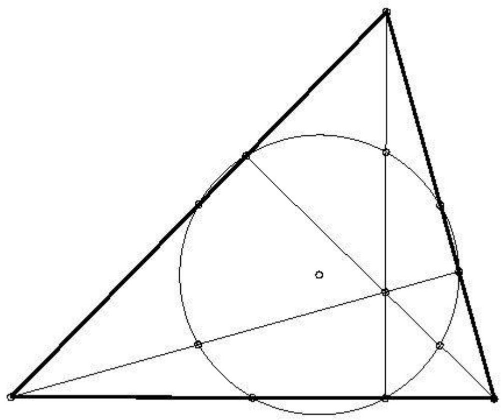

FIGURE 5

[3]) The names "Feuerbach circle" or "Euler circle" are also common.

THEOREM 4.1. *Let $p_1p_2p_3$ be a triangle in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$ with circumcircle $C(x, \lambda)$ and $\mathcal{C}$-orthocenter p. The circle $C\left(\frac{1}{2}(x+p), \frac{1}{2}\lambda\right)$ (called the* Feuerbach circle *of the triangle $p_1p_2p_3$) passes through six "remarkable" points, namely the midpoints of the sides of $p_1p_2p_3$, and the midpoints of the segments $[pp_i]$, $i = 1,2,3$; see Figure 6.*

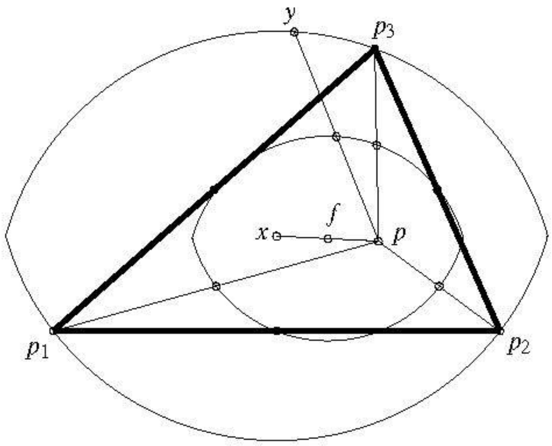

FIGURE 6

REMARK 4.2. In view of (1), the center f of the Feuerbach circle of the triangle $p_1p_2p_3$ satisfies

$$f = \frac{1}{2}(p_1 + p_2 + p_3 - x),$$

where x is the circumcenter of $p_1p_2p_3$.

REMARK 4.3. The Feuerbach circle in any strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$ passes through the three points, each of which is obtained as intersection of a side of the triangle with the line determined by the opposite vertex and the $\mathcal{C}$-orthocenter, if and only if $(\mathbf{A}_2, \mathcal{C})$ is Euclidean. For this reason it is not correct to use the term "nine-point circle" in strictly convex Minkowski planes. So in the sequel the notion of "Feuerbach circle" will be adopted.

REMARK 4.4. Let $\{p_1, p_2, p_3, p_4\}$ be a $\mathcal{C}$-orthocentric system in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, and x_i be the circumcenter of the triangle $p_jp_kp_l$, where $\{i,j,k,l\} = \{1,2,3,4\}$. The triangles $p_jp_kp_l$ have the same Feuerbach circle, whose center is the midpoint of $[p_ix_i]$; see Proposition 2.12.

We will now present some new results on Feuerbach circles of triangles in strictly convex Minkowski planes.

PROPOSITION 4.5. *Let* $\{p_1, p_2, p_3, p_4\}$ *be a* $\mathcal{C}$*-orthocentric system in a strictly convex Minkowski plane* $(\mathbf{A}_2, \mathcal{C})$*, and* x_i *be the circumcenter of the triangle* $p_j p_k p_l$*, where* $\{i, j, k, l\} = \{1, 2, 3, 4\}$*. Then the Feuerbach circles of the triangles* $p_i p_j p_k$ *and* $x_i x_j x_k$ *coincide.*

Proof. This follows immediately from Theorem 4.1 and Corollary 2.11. □

THEOREM 4.6. *Let* $p_1 p_2 p_3$ *be a triangle in a strictly convex Minkowski plane* $(\mathbf{A}_2, \mathcal{C})$ *with circumcircle* $C(x, \lambda)$ *and* $\mathcal{C}$*-orthocenter* p*. If* y *is an arbitrary point on* $C(x, \lambda)$*, then the midpoint of the segment* $[py]$ *lies on the Feuerbach circle of* $p_1 p_2 p_3$ *; see Figure 6.*

Proof. Without loss of generality we may assume that $x \equiv O$. Then the Feuerbach circle of $p_1 p_2 p_3$ is given by

$$C\left(\frac{1}{2}(p_1 + p_2 + p_3), \frac{1}{2}\lambda\right),$$

see Theorem 4.1 and Remark 4.2.

Further, let $y \in C(x, \lambda)$, and m be the midpoint of $[py]$, i.e. $m = \frac{1}{2}(p + y)$. Since $p = p_1 + p_2 + p_3$, see (1), we have

$$\|m - \frac{1}{2}(p_1 + p_2 + p_3)\| = \frac{1}{2}\|y\| = \frac{1}{2}\lambda.$$

This means that

$$m \in C\left(\frac{1}{2}(p_1 + p_2 + p_3), \frac{1}{2}\lambda\right). \qquad \square$$

THEOREM 4.7. *Let* $p_1 p_2 p_3$ *be a triangle in a strictly convex Minkowski plane* $(\mathbf{A}_2, \mathcal{C})$ *with circumcircle* $C(x, \lambda)$*, and let* m_i *be the midpoint of the side* $[p_j p_k]$*, where* $\{i, j, k\} = \{1, 2, 3\}$*. Then the circles* $C\big(m_i, \frac{1}{2}\lambda\big)$ *intersect in the center of the Feuerbach circle of* $p_1 p_2 p_3$ *; see Figure 7.*

Proof. Again we assume, without loss of generality, that $x \equiv O$. If p is the $\mathcal{C}$-orthocenter of $p_1 p_2 p_3$, then $p = p_1 + p_2 + p_3$; see (1). Thus, for the center f of the Feuerbach circle of $p_1 p_2 p_3$ we get

$$f = \frac{1}{2}(p_1 + p_2 + p_3). \tag{7}$$

Therefore

$$\|m_i - f\| = \frac{1}{2}\|p_j + p_k - p_1 - p_2 - p_3\| = \frac{1}{2}\|p_i\| = \frac{1}{2}\lambda. \qquad \square$$

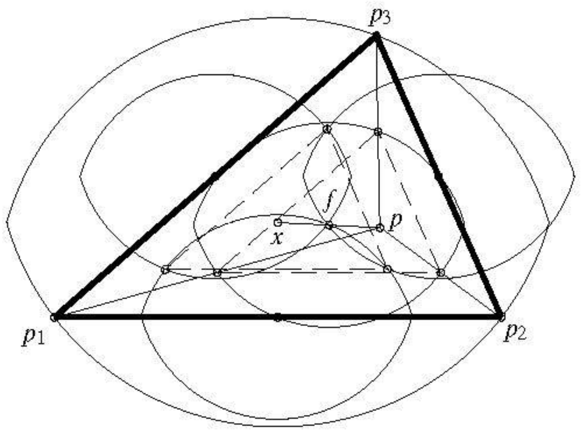

FIGURE 7

THEOREM 4.8. *Let p_1, p_2, p_3, p_4 be four pairwise-distinct points in a strictly convex Minkowski plane $(\mathbf{A}_2, C)$, lying on the circle $C(x, \lambda)$. Then the Feuerbach circles of all four triangles derived from $\{p_1, p_2, p_3, p_4\}$ have a common point p, and their centers lie on the circle $C\big(p, \frac{1}{2}\lambda\big)$; see Figure 8.*

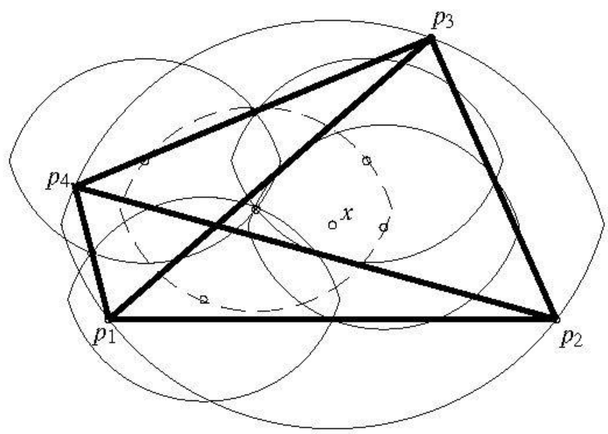

FIGURE 8

Proof. In view of Remark 4.2 the center of the Feuerbach circle of $p_j p_k p_l$ is given by $\frac{1}{2}(p_j + p_k + p_l - x)$. Let

$$p = \frac{1}{2}(p_1 + p_2 + p_3 + p_4) - x\,. \tag{8}$$

Since

$$\|\frac{1}{2}(p_j + p_k + p_l - x) - p\| = \| - \frac{1}{2}p_i + \frac{1}{2}x\| = \frac{1}{2}\lambda, \tag{9}$$

we obtain that the point p lies on the Feuerbach circle of the triangle $p_j p_k p_l$. Then (9) also shows that the center of the Feuerbach circle of $p_j p_k p_l$ lies on the circle $C(p, \frac{1}{2}\lambda)$. □

REMARK 4.9. We call the circle $C(p, \frac{1}{2}\lambda)$ from the above theorem the *Feuerbach circle* of the inscribed quadrangle $p_1 p_2 p_3 p_4$.

REMARK 4.10. Equation (8) implies that the centroid of $p_1 p_2 p_3 p_4$ coincides with the midpoint of the segment between the circumcenter x of $p_1 p_2 p_3 p_4$ and the center p of the Feuerbach circle of $p_1 p_2 p_3 p_4$.

REMARK 4.11. For the Euclidean plane, Theorem 4.8 is a well-known statement; see [19], pp. 22–23, and [14].

It is known that in the Euclidean plane the perpendiculars to the sides of a cyclic quadrangle drawn from the midpoints of the opposite sides are concurrent in the center of the Feuerbach circle of the quadrangle (see [18, p. 146]). The next theorem is an extension of this statement to all strictly convex normed planes.

THEOREM 4.12. *Let $p_1 p_2 p_3 p_4$ be a quadrangle inscribed to the circle $C(x, \lambda)$, and let p be the center of the Feuerbach circle of $p_1 p_2 p_3 p_4$. If p_{ij} is the midpoint of the segment $[p_i p_j]$, then*

$$p - p_{ij} \# \frac{1}{2}(p_k - p_l),$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. Assuming that $x = O$, we have

$$p = \frac{1}{2}(p_1 + p_2 + p_3 + p_4),$$

see (8). Thus we get $p - p_{ij} = \frac{1}{2}(p_k + p_l)$. By $\|p_k\| = \|p_l\|$ we have

$$\frac{1}{2}(p_k + p_l) \# \frac{1}{2}(p_k - p_l),$$

using Lemma 3.1. Therefore $p - p_{ij} \# \frac{1}{2}(p_k - p_l)$. □

We will now prove that the Feuerbach circle of any inscribed quadrangle $p_1p_2p_3p_4$ coincides with the Feuerbach circle of the quadrangle formed by the $\mathcal{C}$-orthocenters of all four triangles $p_ip_jp_k$, where $\{i,j,k,l\} = \{1,2,3,4\}$.

REMARK 4.13. Corollary 2.15 guarantees that the quadrangle formed by the $\mathcal{C}$-orthocenters is also inscribed.

THEOREM 4.14. *Let p_1, p_2, p_3, p_4 be four pairwise-distinct points in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, lying on the circle $C(x,\lambda)$. Let h_i be the $\mathcal{C}$-orthocenter of the triangle $p_jp_kp_l$, where $\{i,j,k,l\} = \{1,2,3,4\}$. Then the Feuerbach circles of the quadrangles $p_1p_2p_3p_4$ and $h_1h_2h_3h_4$ coincide.*

Proof. Assume $x \equiv O$. Then $h_i = p_j + p_k + p_l$, and the Feuerbach circle of $p_1p_2p_3p_4$ is $C\big(p, \frac{1}{2}\lambda\big)$, where

$$p = \frac{1}{2}(p_1 + p_2 + p_3 + p_4),$$

see (8). By Remark 2.7, the circumcenter of the quadrangle $h_1h_2h_3h_4$ is $p_1 + p_2 + p_3 + p_4$. Therefore the center of the Feuerbach circle of $h_1h_2h_3h_4$ is

$$\frac{1}{2}(h_1 + h_2 + h_3 + h_4) - (p_1 + p_2 + p_3 + p_4) = p,$$

see again (8). Moreover, the radius of the circle determined by h_1, h_2, h_3, h_4 is λ, see Corollary 2.15. This means that $C\big(p, \frac{1}{2}\lambda\big)$ is the Feuerbach circle of the quadrangle $h_1h_2h_3h_4$. Thus the proof is done. □

REMARK 4.15. From the equations

$$h_i = p_j + p_k + p_l = p_1 + p_2 + p_3 + p_4 - p_i = 2p - p_i$$

we get $h_i + p_i = 2p$. This implies that the quadrangles $p_1p_2p_3p_4$ and $h_1h_2h_3h_4$ are symmetric with respect to the center of their Feuerbach circle.

The notion of the Feuerbach circle of a quadrangle can be generalized in the following way. Let $p_1p_2\ldots p_n$, $n \geq 2$, be an n-gon inscribed to the circle $C(x,\lambda)$. The circle

$$C\left(\frac{1}{2}\left[\sum_{i=1}^{n} p_i - (n-2)\,x\right], \frac{1}{2}\lambda\right)$$

is called the *Feuerbach circle* of $p_1p_2\ldots p_n$.

REMARK 4.16. According to the above definition, the Feuerbach circle of a chord $[p_1p_2]$ of the circle $C(x,\lambda)$ is the circle $C\big(\frac{1}{2}(p_1+p_2),\frac{1}{2}\lambda\big)$. For $n=3$ this definition coincides with the definition of the Feuerbach circle of a triangle; see Theorem 4.1. For $n=4$ it coincides with the definition given in Remark 4.9.

REMARK 4.17. Obviously, we can rewrite the expression for the center f of the Feuerbach circle of any n-gon inscribed to the circle $C(x,\lambda)$ in the following way:

$$\frac{2}{n}f+\frac{n-2}{n}x=\frac{1}{n}(p_1+p_2+\ldots+p_n).$$

From this we get that the centroid of the considered n-gon divides the segment between the circumcenter and the center of the Feuerbach circle in the ratio $2:(n-2)$.

THEOREM 4.18. *Let $p_1p_2\ldots p_n$, $n\geq 3$, be an n-gon inscribed to the circle $C(x,\lambda)$. Then the Feuerbach circles of all $(n-1)$-gons obtainable from the points $p_1,p_2,\ldots,p_n$ have a common point which is the center of the Feuerbach circle of the n-gon $p_1p_2\ldots p_n$.*

Proof. The number of all $(n-1)$-gons obtainable from $\{p_1,p_2,\ldots,p_n\}$ is n. The center of the Feuerbach circle of the $(n-1)$-gon $p_1\ldots p_{i-1}p_{i+1}\ldots p_n$, $i=1,2,\ldots,n$, is

$$\frac{1}{2}\left[\sum_{s=1}^{n}p_s-p_i-(n-3)\,x\right].$$

The center of the Feuerbach circle of the n-gon $p_1p_2\ldots p_n$ is

$$\frac{1}{2}\left[\sum_{s=1}^{n}p_s-(n-2)\,x\right]. \tag{10}$$

Since

$$\left\|\frac{1}{2}\left[\sum_{s+1}^{n}p_s-p_i-(n-3)x\right]-\frac{1}{2}\left[\sum_{s+1}^{n}p_s-(n-2)x\right]\right\|=\frac{1}{2}\|x-p_i\|=\frac{1}{2}\lambda, \tag{11}$$

all the circles

$$C\left(\frac{1}{2}\left[\sum_{s+1}^{n}p_s-p_i-(n-3)\,x\right],\frac{1}{2}\,\lambda\right)$$

pass through the point (10). And from (11) we obtain the second part of the theorem. □

REMARK 4.19. From (11) we get that the centers of the Feuerbach circles of all $(n-1)$-gons obtainable from the vertices of an n-gon $p_1p_2\dots p_n$ inscribed to $C(x,\lambda)$ lie on the Feuerbach circles of $p_1p_2\dots p_n$.

5. TRIANGLES ASSOCIATED TO A GIVEN TRIANGLE

Let $p_1p_2p_3$ be a triangle with circumcenter $x \equiv O$ in a strictly convex Minkowski plane. In [2] Asplund and Grünbaum considered the triangle whose vertices p_1+p_2, p_2+p_3, p_3+p_1 are obtained by "mirroring" the circumcenter of $p_1p_2p_3$ on the midpoints of its sides. We will call this triangle the *first associated triangle* of $p_1p_2p_3$. It has the following properties (see [2]):

- the circumcenter of the first associated triangle coincides with the $\mathcal{C}$-orthocenter of the original triangle $p_1p_2p_3$;
- the $\mathcal{C}$-orthocenter of the first associated triangle coincides with the circumcenter of $p_1p_2p_3$;
- the Feuerbach circles of any given triangle and its first associated triangle coincide;
- any triangle and its first associated triangle have the same size (i.e., the lengths of their sides are correspondingly equal).

We will now consider two more triangles associated to an arbitrary triangle in a strictly convex Minkowski plane.

PROPOSITION 5.1. *Let $p_1p_2p_3$ be a triangle with $\mathcal{C}$-orthocenter in a strictly convex Minkowski plane $(\mathbf{A}_2,\mathcal{C})$, and let m_i be the midpoint of the side $[p_jp_k]$, where $\{i,j,k\}=\{1,2,3\}$. If*

$$q_i = C\big(m_j, \tfrac{1}{2}\lambda\big) \cap C\big(m_k, \tfrac{1}{2}\lambda\big),$$

then the points q_1,q_2,q_3 and the center f of the Feuerbach circle of $p_1p_2p_3$ form an orthocentric system, and the relations

$$(12) \qquad q_i - q_j = \frac{1}{2}(p_i - p_j), \qquad q_i - f = \frac{1}{2}(p_i - p)$$

hold; see Figure 7.

Proof. In order to prove the first part of the statement, it is sufficient (in view of Theorem 4.7) to observe that the circumradius of $q_1q_2q_3$ equals $\frac{1}{2}\lambda$. By Lemma 2.2 and Theorem 4.7 we have

$$(13) \qquad q_i + f = m_j + m_k\,.$$

Assuming that the circumcenter of $p_1p_2p_3$ coincides with the origin O and using (7) and (13), we get

$$\|q_i\| = \|m_j + m_k - f\| = \frac{1}{2}\,\|(p_i+p_k)+(p_i+p_j)-(p_1+p_2+p_3)\| = \frac{1}{2}\,\|p_i\| = \frac{1}{2}\,\lambda\,.$$

Furthermore, the relations (12) follow immediately from (7) and (13). □

REMARK 5.2. We call the triangle $q_1q_2q_3$ the *second associated triangle* of $p_1p_2p_3$.

REMARK 5.3. The proof of Proposition 5.1 implies that the circumcenters of $p_1p_2p_3$ and $q_1q_2q_3$ coincide.

COROLLARY 5.4. *Let $p_1p_2p_3$ be a triangle in a strictly convex Minkowski plane $(\mathbf{A}_2, \mathcal{C})$, having the $\mathcal{C}$-orthocenter p and the Feuerbach circle $C(f, \frac{1}{2}\lambda)$. If*

$$r_i = C\big(f, \frac{1}{2}\,\lambda\big) \cap \langle pp_i \rangle\,,$$

then the points r_1, r_2, r_3, p form an orthocentric system and the relations

$$r_i - r_j = \frac{1}{2}\,(p_i - p_j)\,, \qquad r_i - p = \frac{1}{2}\,(p_i - p)$$

hold; see Figure 7.

Proof. We only have to note that r_i is the midpoint of $[p_ip]$ (see Theorem 4.1), since the proof of this statement is quite similar to that of Proposition 5.1. □

REMARK 5.5. We call $r_1r_2r_3$ the *third associated triangle* of $p_1p_2p_3$.

REMARK 5.6. The circumcenter of the triangle $r_1r_2r_3$ coincides with the center of the Feuerbach circle of $p_1p_2p_3$.

In analogy to the observations of Asplund and Grünbaum, we summarize the properties of the second and of the third associated triangle:

- the circumcenters of any triangle and its second associated triangle coincide;
- the $\mathcal{C}$-orthocenters of any triangle and its third associated triangle coincide;
- the center of the Feuerbach circle of any triangle coincides with the $\mathcal{C}$-orthocenter of its second associated triangle and with the circumcenter of the third associated triangle;

• the Feuerbach circles of the second and third associated triangle of any triangle are symmetric with respect to the center of the Feuerbach circle of the original triangle.

6. ON THE EIGHT-POINT CIRCLE OF QUADRANGLES

The nine-point circle is concerned with a triangle. But quadrangles of a special type also possess such a circle, which passes through the midpoints of their sides. The following theorem, called the *eight-point circle theorem*, describes this analogue in the Euclidean plane. For the proof we refer to [4].

THEOREM 6.1. *Let $abcd$ be a quadrangle with perpendicular diagonals ac and bd in the Euclidean plane. Let p, q, r, s be the midpoints of the sides ab, bc, cd, and da, respectively. Let p' be the foot-point of the perpendicular line from r to the opposite quadrangle side, and q', r', s' be defined in the same way. Then the points $p, q, r, s, p', q', r', s'$ lie on the same circle; see Figure 9.*

REMARK 6.2. The circle determined by $p, q, r, s, p', q', r', s'$ is called the *eight-point circle* of the quadrangle $abcd$.

Theorem 6.1 on the eight-point circle can be extended in some way to *all* Minkowski planes.

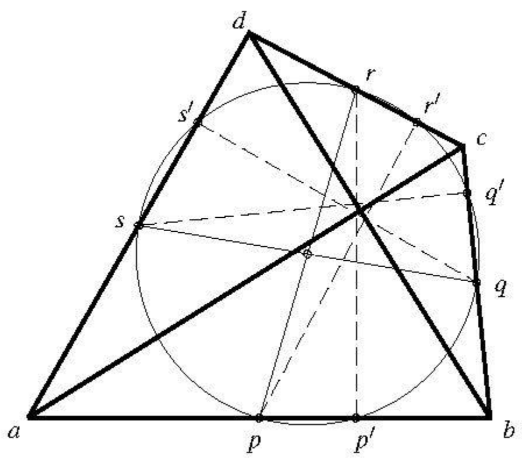

FIGURE 9

LEMMA 6.3. *In an arbitrary Minkowski plane, let abcd be a quadrangle whose diagonals* $a-c$ *and* $b-d$ *are James orthogonal. If* p,q,r,s *are the midpoints of the sides* $[ab]$, $[bc]$, $[cd]$ *and* $[da]$, *respectively, then* p,q,r,s *lie on the same circle.*

Proof. Since $p-q=s-r=\frac{1}{2}(a-c)$ and $p-s=q-r=\frac{1}{2}(b-d)$, the quadrangle $pqrs$ is a parallelogram with $p-q \,\#\, p-s$. James orthogonality of $p-q$ and $p-s$ means that $\|p-r\|=\|s-q\|$, and if $m=\langle pr\rangle\cap\langle sq\rangle$, then

$$\|m-p\|=\|m-q\|=\|m-r\|=\|m-s\|.$$

Therefore the points p,q,r,s lie on a circle with center m. □

The configuration described by Lemma 6.3 is shown in Figure 10 for the rectilinear norm, which is neither strictly convex nor smooth.

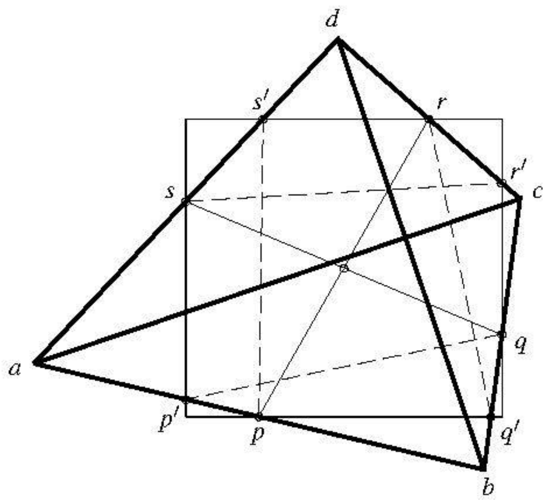

FIGURE 10

Also the next statement holds for all Minkowski planes.

THEOREM 6.4. *In an arbitrary Minkowski plane let abcd be a quadrangle whose diagonals* $a-c$ *and* $b-d$ *are James orthogonal. Let* p,q,r,s *be the midpoints of the sides* $[ab]$, $[bc]$, $[cd]$ *and* $[da]$, *respectively, and* $C(m,\lambda)$ *be the circle determined by these four points. If* $p'\in C(m,\lambda)\cap\langle ab\rangle$, $q'\in C(m,\lambda)\cap\langle bc\rangle$, $r'\in C(m,\lambda)\cap\langle cd\rangle$, $s'\in C(m,\lambda)\cap\langle da\rangle$, *then*

$$p-p' \,\#\, p'-r\,,\quad q-q' \,\#\, q'-s\,,\quad r-r' \,\#\, r'-p\,,\quad s-s' \,\#\, s'-q\,. \tag{14}$$

Proof. By Lemma 6.3, m is the intersection of the diagonals of the parallelogram $pqrs$. Since $p, p', r \in C(m, \lambda)$ and since m is the midpoint of $[pr]$, we have

$$\|m - p'\| = \frac{1}{2}\|p - r\|.$$

This means that the diagonals of the parallelogram spanned by $p - p'$ and $p' - r$ have the same length, i.e., $p - p' \# p' - r$. The remaining relations in (14) can be proven in the same way. □

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