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THE HOMOTOPY EXPONENT CONJECTURE FOR p-COMPLETED CLASSIFYING SPACES OF FINITE GROUPS

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CONJECTURE 25.1. *Let π be a finite group, and let $B\pi_p^\wedge$ denote its p-completed classifying space. Then the homotopy groups of $B\pi_p^\wedge$ have a bounded exponent, i.e., there exists an integer r such that*

$$p^r \cdot \pi_*(B\pi_p^\wedge) = \{0\}\,.$$

Since π is finite, the fundamental group of $B\pi_p^\wedge$ is the finite p-group given by the quotient of π by its minimal normal subgroup of p-power index, denoted by $O^p(\pi)$. If the order of $O^p(\pi)$ is not divisible by p, then $B\pi_p^\wedge$ is homotopy equivalent to $B(\pi/O^p(\pi))$, and the conjecture reduces to a triviality. If p divides the order of $O^p(\pi)$, then $B\pi_p^\wedge$ has infinitely many non-vanishing homotopy groups, all of which are finite p-groups. It is therefore natural to ask whether there is an upper bound on the exponent of these homotopy groups.

Our conjecture is related to the Moore finite exponent conjecture. Recall that a group G is said to have exponent m if every element $x \in G$ of finite order has order dividing m. The Moore conjecture states that for a simply-connected space X of the homotopy type of a finite CW-complex, the graded group $\pi_*(X) \otimes \mathbf{Q}$ is a finite dimensional rational vector space if and only if for every prime p, the graded group $\pi_*(X) \otimes \mathbf{Z}_{(p)}$ has an exponent.

In particular, if X is a finite simply connected CW-complex whose homotopy groups are all finite, then the Moore conjecture implies that $\pi_*(X)$ has an exponent. One can show that if π is a finite group, then the component of the constant loop in $\Omega B\pi_p^\wedge$ is a retract of the loop space of a finite simply connected torsion complex. Therefore our conjecture would follow at once if the much stronger Moore conjecture were true.

For any finite group π, $\pi_i(B\pi_p^\wedge) \cong \pi_i(BO^p(\pi)_p^\wedge)$ for all $i \geq 2$. Furthermore, for each $i \geq 3$ $\pi_i(B\pi_p^\wedge) \cong \pi_i(BU^p(\pi)_p^\wedge)$, where $U^p(\pi)$ is the p-universal central extension of $O^p(\pi)$. In all known examples for the conjecture, the order of the Sylow p-subgroup of $U^p(\pi)$ is an upper bound for the order of torsion in $\pi_*(B\pi_p^\wedge)$. There are examples where this bound is sharp.

It is known that for any finite group π, the p-torsion in the homology of the loop space $\Omega B\pi_p^\wedge$ is bounded above by the order of the Sylow p-subgroup of $O^p(\pi)$.

EXAMPLE 25.2. Some examples are known. A few are given by

$$\pi = A_5, A_6, A_7, J_1, M_{11}$$

at $p = 2$. A few more at the prime 2 are given by those finite simple groups of 2-rank 2 (including M_{11}) with the possible exception of $U(3, \mathbf{F}_4)$. The finite simple groups of classical Lie type over the field $\mathbf{F}_q$, where q is a power of a prime different from p, provide a large family of examples at the prime p.

Finite simple groups of Lie type at the defining characteristic. Almost no examples are known of the behavior of $\pi_i(BG(\mathbf{F}_{p^k})_p^\wedge)$, where G is a finite simple groups of Lie type.

Finite groups with Abelian Sylow p-subgroups. For a finite group π with cyclic Sylow p-subgroup and $p > 2$, $B\pi_p^\wedge$ is known to have a homotopy exponent. Furthermore, the best possible upper bound for this exponent is given by the order of the Sylow p-subgroup. On the other hand, for a finite group π with an abelian Sylow p-subgroup of rank larger than 1 neither one of the above statements is known to hold.

Alternating groups. Further natural open cases are A_n with $n > 7$ at the prime 2, and at all primes p in the cases where the Sylow p-subgroup is not cyclic.

The examples mentioned above are obtained by considering the structure of the loop space of $B\pi_p^\wedge$. This space sometimes admits a non-trivial product decomposition or is finitely resolvable by fibrations involving more recognizable spaces, which are known to have homotopy exponents by the work of Cohen–Moore–Neisendorfer [2] and [3].

The largest and most comprehensive reference on this subject is [4], where some other theoretical results are established, and many examples are given.

This paper is motivated by the spherical resolvability conjecture for $\Omega BG_p^\wedge$ due to the first author. This conjecture, if it was true would imply the conjecture proposed here. Unfortunately, as demonstrated in [5], the spherical resolvability conjecture is false, but at the same time some of the examples that prove it wrong do satisfy the finite exponent conjecture ([1], Cor. 5.4). A survey of many of the known results on spaces of type $B\pi_p^\wedge$ for π finite, including examples of homotopy exponents, is the paper [1].

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