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## 27

### THE HOPF CONJECTURE AND THE SINGER CONJECTURE

by Michael W. DAVIS

CONJECTURE 27.1. *Suppose  $M^{2k}$  is a closed, aspherical manifold of dimension  $2k$ . Then  $(-1)^k \chi(M^{2k}) \geq 0$ .*

The conjecture is true in dimension 2 since the only surfaces which have positive Euler characteristic are  $S^2$  and  $\mathbf{RP}^2$  and they are the only two which are not aspherical. In the special case where  $M^{2k}$  is a nonpositively curved Riemannian manifold this conjecture is usually attributed to Hopf by topologists and either to Chern or to both Chern and Hopf by differential geometers.

When I first heard about this conjecture in 1981, I thought I could come up with a counterexample by using right-angled Coxeter groups. Given a finite simplicial complex  $L$  which is a flag complex, there is an associated right-angled Coxeter group  $W$ . Its Euler characteristic is given by the formula

$$(27.1) \quad \chi(W) = 1 + \sum_{i=0}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i,$$

where  $f_i$  denotes the number of  $i$ -simplices in  $L$ . If  $L$  is a triangulation of  $S^{n-1}$ , then  $W$  acts properly and cocompactly on a contractible  $n$ -manifold. The quotient of this contractible manifold by any finite index, torsion-free subgroup  $\Gamma \subset W$  is a closed aspherical  $n$ -manifold  $M^n$ . Since  $\chi(M^n)$  is a positive multiple of  $\chi(W)$  (by  $[W : \Gamma]$ ), they have the same sign. So, this looked like a good way to come up with counterexamples to Conjecture 27.1. Conversely, if you believe Conjecture 27.1, then you must also believe the following

CONJECTURE 27.2. *If  $L$  is any flag triangulation of  $S^{2k-1}$  then*

$$(-1)^k \kappa(L) \geq 0,$$

where  $\kappa(L)$  is the quantity defined by the right-hand side of (27.1).

Ruth Charney and I published this conjecture in [2]. It is sometimes called the Charney–Davis Conjecture.

In the 1970’s Atiyah [1] introduced  $L^2$  methods into topology. If a discrete group  $\Gamma$  acts properly and cocompactly on a smooth manifold or a CW-complex  $Y$ , then one can define the reduced  $L^2$ -cohomology spaces of  $Y$  and their “dimensions” with respect to  $\Gamma$ , the so-called “ $L^2$ -Betti numbers”. Let  $L^2 b_i(Y; \Gamma)$  be the  $\Gamma$ -dimension of the  $L^2$ -cohomology of  $Y$  in dimension  $i$ . It is a *nonnegative* real number. If  $Y \rightarrow X$  is a regular covering of a finite CW-complex  $X$  with group of deck transformations  $\Gamma$ , the Euler characteristic of  $X$  can be calculated from the  $L^2$ -Betti numbers of  $Y$  by the formula

$$(27.2) \quad \chi(X) = \sum (-1)^i L^2 b_i(Y; \Gamma).$$

Shortly after Atiyah described this formula in [1], Dodziuk [4] and Singer realized that there is a conjecture about  $L^2$ -Betti numbers which is stronger than Conjecture 27.1. It is usually called the Singer Conjecture. Beno Eckmann [5] also discusses it in this volume.

CONJECTURE 27.3 ([4]). *Suppose  $M^n$  is a closed, aspherical manifold with fundamental group  $\pi$  and universal cover  $\widetilde{M}^n$ . Then  $L^2 b_i(\widetilde{M}^n; \pi) = 0$  for all  $i \neq \frac{n}{2}$ . (In particular, when  $n$  is odd this means all its  $L^2$ -Betti numbers vanish.)*

This implies Conjecture 27.1 since, when  $n = 2k$ , formula (27.2) gives:  $(-1)^k \chi(M^{2k}) = L^2 b_k(\widetilde{M}^{2k}; \pi) \geq 0$ .

Of course, there is also the following version of Conjecture 27.3 for Coxeter groups.

CONJECTURE 27.4. *Suppose that  $L$  is a triangulation of  $S^{n-1}$  as a flag complex, that  $W$  is the associated right-angled Coxeter group and that  $\Sigma$  is the contractible  $n$ -manifold on which  $W$  acts. Then  $L^2 b_i(\Sigma; W) = 0$  for all  $i \neq \frac{n}{2}$ .*

Boris Okun and I discussed this conjecture in [3] and we proved it for  $n \leq 4$ . The result for  $n = 4$  implies Conjecture 27.2 when  $L$  is a flag triangulation of  $S^3$ . So, Conjecture 27.2 is true in the first dimension for which it is not obvious.

## REFERENCES

- [1] ATIYAH, M. F. Elliptic operators, discrete groups and von Neumann algebras. In: *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, 43–72. Astérisque 32–33. Soc. Math. France, 1976.
- [2] CHARNEY, R. and M. W. DAVIS. The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold. *Pacific J. Math.* 171 (1995), 117–137.
- [3] DAVIS, M. W. and B. OKUN. Vanishing theorems and conjectures for the  $L^2$ -homology of right-angled Coxeter groups. *Geom. Topol.* 5 (2001), 7–74.
- [4] DODZIUK, J.  $L^2$  harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. Amer. Math. Soc.* 77 (1979), 395–400.
- [5] ECKMANN, B. The Singer conjecture. (*This volume.*)

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