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PIECEWISE ISOMETRIES OF HYPERBOLIC SURFACES

by Pierre DE LA HARPE

What does the group of piecewise isometries of a surface look like ?

More precisely, let us consider compact Riemannian surfaces. Boundaries (if any) should be unions of finitely many geodesic segments; there is no reason to impose connectedness or orientability. For two surfaces M, N of this kind, a *piecewise isometry* from M to N is given by two partitions $M = \bigsqcup_{i=1}^{k} M_i$ and $N = \bigsqcup_{i=1}^{k} N_i$ in polygons, and a family $g_i \colon M_i \longrightarrow N_i$ of surjective isometries; two such piecewise isometries are identified if they coincide on the interiors of the pieces of finer polygonal partitions. When such a piecewise isometry exists, M and N are said to be *equidecomposable*. Piecewise isometries of a surface M to itself constitute the *group of piecewise isometries* $\mathcal{PI}(M)$. We want to stress that a piecewise isometry need not be continuous. The group $\mathcal{PI}(M)$ is a two-dimensional analogue of the group $\mathcal{PI}([0,1])$ of exchange transformations of the interval (the transformations themselves have been studied by Keane, Sinai, and Veech, among others, and the group by Arnoux, Fathi, and Sah — see for example [7] and [1]).

It is well known that two Euclidean polygons are equidecomposable if and only if their areas are equal (compare with Chapter IV in Hilbert's *Grundlagen der Geometrie* [9]). This carries over to polygons in the hyperbolic plane (see [4] for a proof). In particular, any orientable connected closed Riemannian surface M of genus $g \geq 2$ and of constant curvature -1 is piecewise isometric to a hyperbolic polygon, of area $4\pi(g-1)$. Thus, viewed as an abstract group, $\mathcal{PI}(M)$ depends only on the area t of M, and can be denoted by $\mathcal{PI}_t$. There are many ways to check that it is an uncountable group, containing torsion of any order and containing free abelian groups of arbitrary large ranks. Observe that, if $s \leq t$, the group $\mathcal{PI}_s$ embeds as a subgroup of $\mathcal{PI}_t$ (think of a hyperbolic polygon of area s contained inside a hyperbolic polygon of area t).

I would like to understand more of the groups $\mathcal{PI}_t$.

As a first question, are these groups pairwise isomorphic? In particular, are $\mathcal{PI}_{4\pi}$ and $\mathcal{PI}_{8\pi}$ isomorphic? (Recall that, for a Riemannian metric of constant curvature -1, a closed surface of genus g has area $4\pi(g-1)$.) If $s \leq t$, is any injective homomorphism $\mathcal{PI}_s \to \mathcal{PI}_t$ conjugate to one described above?

Are these groups acyclic? Simple? Or if not with simple commutator subgroups? (Arnoux-Fathi and Sah have defined a homomorphism from the analogous group $\mathcal{PI}([0,1])$ onto $\bigwedge_{\mathbf{Q}}^2 \mathbf{R}$, reminiscent of the Dehn invariant for scissors congruences, and it is known that the kernel is a simple group; see [1]).

Should they be regarded as topological groups? If yes for which topology? (Two candidates: the topology of convergence in measure, see e.g. [3], and the weak topology discussed in [8].)

Similar questions make sense for other groups of piecewise isometries, for example related to polygons in a round sphere, or in a flat torus, or related to other spaces and appropriates pieces. The case of flat tori is usually phrased in terms of Euclidean spaces or polytopes; concerning this case, the little I am aware of ([2], [5], [10]) is about particular piecewise isometries and not about groups $\mathcal{PI}(M)$. One difficulty with other spaces is to choose an interesting class of pieces when "polygon" or "polytope" have no clear meaning.

A bijection of a finitely-generated group onto a subset of itself which is given piecewise by left multiplications can be viewed as a piecewise isometry. Bijections of this form are important ingredients in the theory of amenable groups (Tarski characterization of non-amenability by the existence of paradoxical decompositions, see e.g. [11] and [6]).

Piecewise isometries make sense for large classes of metric spaces, but the corresponding groups and pseudogroups seem to have been little explored so far in this generality.

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