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EXOTIC CLASSES IN THE MOD 2 COHOMOLOGY OF $\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}])$

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Let D_n denote the subgroup of the algebraic group GL_n consisting of diagonal matrices; let ι_n denote the inclusion of discrete groups $\mathrm{D}_n(\mathbf{Z}[\frac{1}{2}]) \hookrightarrow \mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}])$.

S. Mitchell [5] has shown that the image of the restriction map

$$\iota_n^*\colon\ \mathrm{H}^*\big(\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}]);\mathbf{F}_2\big) \to \mathrm{H}^*\big(\mathrm{D}_n(\mathbf{Z}[\frac{1}{2}]);\mathbf{F}_2\big)$$

is isomorphic to $\mathbf{F}_2[w_1,w_2,\dots,w_n]\otimes\Lambda(e_1,e_3,\dots,e_{2n-1})$. Here the w_i are the *Stiefel–Whitney classes* of the tautological representation $\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}]) \to \mathrm{GL}_n(\mathbf{R})$ and the e_{2i-1} are closely related to Quillen's odd-dimensional *modular characteristic classes* in the mod 2 cohomology of $\mathrm{GL}_n(\mathbf{F}_3)$.

By explicit calculation the map ι_n^* is known to be injective for $n \leq 3$ ([5] for $n=2$ and [3] for $n=3$). Work of Dwyer [1] shows that ι_n^* fails to be injective if $n \geq 32$ and still unpublished work of ours [4] — which will hopefully see the light of the day sometime soon — shows that it already fails if $n \geq 14$. However, the reasoning in [1] as well as in [4] is very indirect and does not produce any explicit elements in the kernel of ι_n^*. On the other hand one of us has shown [2] that the kernel of ι_n^* becomes very large as n grows.

PROBLEM 38.1. Construct explicit elements in the kernel of ι_n^*.

COMMENTS. (1) Injectivity of ι_n^* amounts to the validity of an unstable Lichtenbaum–Quillen conjecture for $\mathbf{Z}[\frac{1}{2}]$ at the prime 2 (cf. [1]) so any (even partial) progress on this problem would shed more light on why such an unstable conjecture fails.

(2) The solution of the Milnor conjecture by Voevodsky led to the proof that the map ι_∞^* (ι_∞ being the colimit of the ι_n) is injective: in other words, the stable Lichtenbaum-Quillen conjecture for $\mathbf{Z}[\frac{1}{2}]$ at the prime 2 holds.

(3) Now let us come back to the ghost reference [4]. Let O_n be the subgroup of GL_n consisting of orthogonal matrices (orthogonal for the euclidian metric) and ρ_n be the homomorphism of discrete groups $\mathrm{O}_n(\mathbf{Z}[\frac{1}{2}]) \to \mathrm{O}_n(\mathbf{F}_3)$ induced by mod 3 reduction. The reasoning in [4] is as follows: first we show that if ι_n^* is injective then ρ_n^* is bijective and afterwards we prove that ρ_n^* is not bijective for $n \geq 15$. (We use a variation of this argument to show that even ι_{14}^* is not injective.)

QUESTION 38.2. Is ρ_∞^* bijective?

A positive answer to this question amounts to the validity of an "orthogonal Lichtenbaum–Quillen conjecture for $\mathbf{Z}[\frac{1}{2}]$ at the prime 2".

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