A conjecture in number theory

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A CONJECTURE IN NUMBER THEORY

by Peter HILTON

Given positive integers b, a, odd, $a < \frac{b}{2}$, $\gcd(a,b) = 1$, we construct a *symbol* or *coach*

$$b\left|\begin{matrix} a_1 a_2 \cdots a_r \\ k_1 k_2 \cdots k_r \end{matrix}\right|, \quad a_1 = a, \tag{39.3}$$

where $b - a_i = 2^{k_i} a_{i+1}$, with k_i maximal positive ($i = 1, 2, \dots, r$), $a_{r+1} = a_1$. Then each a_i is odd with $a_i < \frac{b}{2}$. It is, moreover, known (see Chapter 4 of [1], [2]) that, if $k = \sum_i k_i$, then k is the *quasi-order* of 2 mod b, that is, k is the smallest positive integer such that $2^k \equiv \pm 1 \mod b$. In fact,

$$2^k \equiv (-1)^r \mod b. \tag{39.4}$$

As indicated, if S is the set of all positive integers satisfying the conditions on a above, then $a_i \in S$, $1 \leq i \leq r$. It is possible that the set $\{a_i, 1 \leq i \leq r\}$ exhausts S. If not we may, of course, construct further coaches based on b. For example, with $b = 65$, there are 4 coaches

$$65\left|\begin{matrix} 1 \\ 6 \end{matrix}\right|\begin{matrix} 3 & 31 & 17 \\ 1 & 1 & 4 \end{matrix}\left|\begin{matrix} 7 & 29 & 9 \\ 1 & 2 & 3 \end{matrix}\right|\begin{matrix} 11 & 27 & 19 & 23 & 21 \\ 1 & 1 & 1 & 1 & 2 \end{matrix}\right|, \tag{39.5}$$

forming what we call the *complete symbol* for $b = 65$. We write $c = c(b)$ for the number of coaches in a complete symbol.

We conjecture that it should be possible to determine if a complete symbol has only one coach ($c = 1$), or, more generally, to determine the number of coaches, without having to construct the coaches — and thus determining the quasi-order of 2 mod b. It is, moreover, clear that the theory can be extended to arbitrary bases t and is not confined to the case $t = 2$.

REMARK 39.1. In fact, we know that $\Phi(b) = 2ck$, where Φ is the Euler totient function. Thus $c = 1$ if, and only if, $k = \frac{1}{2}\Phi(b)$. The rule

$$\Phi(b) = 2ck \tag{39.6}$$

is proved in [3], which is the contribution of Peter Hilton, Jean Pedersen and Byron Walden to a Festschrift for Martin Gardner.

REFERENCES

[1] HILTON, P., D. HOLTON and J. PEDERSEN. *Mathematical Reflections. In a Room with Many Mirrors.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1997.

[2] HILTON, P., D. HOLTON and J. PEDERSEN. *Mathematical Vistas. From a Room with Many Windows.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2002.

[3] HILTON, P., J. PEDERSEN and B. WALDEN. A property of complete symbols. In: *Festschrift for Martin Gardner*, to appear.

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