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ALPERIN'S WEIGHT CONJECTURE

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Let p be a prime number.

CONJECTURE 42.1. *Let G be a finite group and let P be a Sylow p-subgroup of G.*

(i) *The number of conjugacy classes of p'-elements of G is greater than or equal to the number of conjugacy classes of $N_G(P)/P$.*

(ii) *If P is abelian, then the number of conjugacy classes of G is greater than or equal to the number of conjugacy classes of $N_G(P)$.*

The above inequalities would follow from Alperin's weight conjecture [1] which we describe now.

Let k be an algebraically closed field of characteristic p. For a finite group H denote by $l(kH)$ the number of isomorphism classes of simple kH-modules, and by $w(kH)$ the number of isomorphism classes of simple projective kH-modules. The weight conjecture predicts the following

CONJECTURE 42.2. *Let G be a finite group. Then*

$$l(kG) = \sum_{Q \in \mathcal{I}} w(kN_G(Q)/Q)\,,$$

where $\mathcal{I}$ denotes a set of representatives of G-conjugacy classes of p-subgroups of G.

Conjecture 42.2 comes in a block-wise version as well. The reformulation of this conjecture in terms of alternating chains in [13] paved the way for many extensions (see for instance [9], [10], [11], [16]). Despite having been verified for many families of finite groups, including finite p-solvable groups, symmetric groups, and in some cases for finite groups of Lie type and some sporadic simple groups (see for instance [2], [3], [4], [5], [6], [8], [12]), a true understanding of Conjecture 42.2 or indeed of Conjecture 42.1 remains elusive.

In its original form stated above, Alperin's weight conjecture is a numerical equality interpreting the number of simple modules of a finite group or a p-block in terms of the involved p-local structure. In subsequent years structural approaches to this and related conjectures in terms of linear source modules [7], fusion category algebras [14], and cohomological invariants of functors over certain finite categories [15], have emerged.

We briefly explain how 42.1 would follow from 42.2. By a theorem of Brauer, $l(kG)$ is equal to the number of conjugacy classes of p'-elements in G, and the summand for $Q = P$ on the right side of the equality 42.2 is equal to the number of conjugacy classes of $N_G(P)/P$. Thus 42.2 implies the inequality 42.1 (i). The inequality 42.1 (ii) follows from 42.1 (i) applied to centralizers of p-elements.

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