Outer automorphisms of order 3 of Lie algebra of type D4

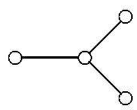
L'Enseignement Mathématique (2) 54 (2008), 126–128

44

OUTER AUTOMORPHISMS OF ORDER 3 OF LIE ALGEBRAS OF TYPE D_4

by Max-Albert KNUS

Simple Lie algebras over algebraically closed fields of characteristic zero are classified by their Dynkin diagrams. Moreover the group of automorphisms of the Lie algebra modulo the subgroup of inner automorphisms is isomorphic to the group of symmetries of the corresponding Dynkin diagram. In most cases this group of symmetries has at most two elements. The case of the Lie algebra of skew-symmetric 8×8-matrices is exceptional. The Dynkin diagram is D_4:



and has the permutation group S_3 as a group of automorphisms. The automorphisms of the Dynkin diagram can easily be extended to automorphisms of the Lie algebra using the root system. Thus over an algebraically closed field, the classes of automorphisms modulo inner automorphisms are explicitly known.

A complete list of conjugacy classes of outer automorphisms of order 3 over an algebraically closed field can be deduced from the classification of automorphisms of finite order of simple Lie algebras (see [3] or [1]). Besides the conjugacy class of the automorphism constructed with help of the root system, whose fixed-point algebra is of type G_2, there is one more conjugacy class in the full group of automorphisms, whose fixed-point algebra is a simple Lie algebra of type A_2.

We consider outer automorphisms of order 3 of simple Lie algebras over an arbitrary field of characteristic zero. The orthogonal Lie algebra relative to the quadratic norm form of a Cayley algebra always admits such automorphisms.

This is known as the "local triality principle". The converse also holds: if the orthogonal Lie algebra relative to a quadratic form admits an outer automorphism of order 3, then the quadratic form is the norm form of a Cayley algebra. Thus triality and octonions are mutually "responsible" (Tits) for existence. By descent, conjugacy classes of order 3 outer automorphisms must have Lie algebras of type G_2 or A_2 as fixed-point algebras. We conjecture that conjugacy classes are essentially classified by the corresponding fixed-point algebras and we give a complete list of candidates. The main ingredient is the notion of a 8-dimensional symmetric composition:

Let S be a finite dimensional F-vector space with a bilinear multiplication $(x,y) \mapsto x \star y$. We say that a quadratic form n on S is *multiplicative* if $n(x \star y) = n(x)n(y)$ holds for all x, $y \in S$. A nonsingular multiplicative quadratic form can only occur in dimension $1,2,4$ and 8. A triple $(S,\star,n)$ with a nonsingular multiplicative quadratic form n is called a *symmetric composition* (see [2]) if the polar b of the norm form n satisfies the relation $b(x \star y, z) = b(x, y \star z)$ for x, z, $y \in S$. The norm form n of a 8-dimensional symmetric composition is always the norm of a (unique) Cayley algebra. However the multiplication of a Cayley algebra does not satisfy the axioms of a symmetric composition. There are two types of symmetric compositions in dimension 8:

Type G_2: Let C be a Cayley algebra with involution $x \mapsto \overline{x}$. The new multiplication $(x,y) \mapsto \overline{x} \cdot \overline{y}$ defines the structure of a symmetric composition on C.

Type A_2: This type is associated with a central simple algebra B of dimension 9 with an involution τ of second kind over the quadratic extension $K = F(\sqrt{-3})$. Let $\mathrm{Sym}(B,\tau)$ be the set of symmetric elements in B and let $\mathrm{Sym}(B,\tau)^0 = \{x \in \mathrm{Sym}(B,\tau) \mid T_B(x) = 0\}$ be the 8-dimensional subspace of reduced trace 0 elements. We define a multiplication $\star$ on $\mathrm{Sym}(B,\tau)^0$ by

$$x \star y = \mu xy + (1-\mu)yx - \tfrac{1}{3}T_B(yx)1\,.$$

where $\mu = \frac{1+\sqrt{-3}}{6}$. Then $\big(\mathrm{Sym}(B,\tau)^0, \star\big)$ is a symmetric composition with norm $n(x) = \frac{1}{6}T_B(x^2)$.

Let $\mathfrak{o}(n) \subset \mathrm{End}_F(S)$ be the orthogonal Lie algebra associated to the norm n of a symmetric composition. For any $f \in \mathfrak{o}(n)$ there are unique elements $g, h \in \mathfrak{o}(n)$ such that

$$f(x \star y) = g(x) \star y + x \star h(y)$$

and $\rho\colon f \mapsto g$, $\rho'\colon f \mapsto h$ are outer automorphisms of order 3 of $\mathfrak{o}(n)$ such that $\rho^2 = \rho'$. Moreover the fixed-point Lie algebra under ρ is the Lie algebra of derivations of the algebra $(S, \star)$. This Lie algebra is of type G_2 if S is of type G_2 and of type A_2 if S is of type A_2 (see [2]). We believe that this construction, via symmetric compositions, gives a complete set of representatives of outer automorphisms of order 3 of orthogonal Lie algebras up to conjugation. The construction of outer automorphisms of order 3 of arbitrary Lie algebras of type D_4 is open.

REFERENCES

[1] HELGASON, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Pure and Applied Mathematics *80*. Academic Press, 1978.

[2] KNUS, M.-A., A. S. MERKURJEV, M. ROST and J.-P. TIGNOL. *The Book of Involutions*. Amer. Math. Soc. Colloquium Publications *44*. Amer. Math. Soc., Providence, R.I., 1998.

[3] WOLF, J. A. and A. GRAY. Homogeneous spaces defined by Lie group automorphisms. I. *J. Differential Geom. 2* (1968), 77–114.

Max-Albert Knus

Mathematik Departement
ETH Zürich
CH–8092 Zürich
Switzerland
e-mail: knus@math.ethz.ch