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### THE KROPHOLLER CONJECTURE

by Graham A. NIBLO and Michah SAGEEV

A finitely generated group  $G$  is said to *split over a subgroup*  $H$  if and only if  $G$  may be decomposed as an amalgamated free product  $G = A *_H B$  (with  $A \neq H \neq B$ ) or as an HNN extension  $G = A *_H$ . The Kropholler conjecture is concerned with the existence of such splittings.

Given a subgroup  $H$  of a finitely generated group  $G$  the *invariant*  $e(G, H)$  is defined to be the number of *Freudenthal (topological) ends* of the quotient of the Cayley graph of  $G$  under the action of the subgroup  $H$ . This number does not depend on the (finite) generating set chosen for  $G$  (see [3]) so it is an invariant of the pair  $(G, H)$ .

For example, if  $G$  is a free abelian group and  $H$  is an infinite cyclic subgroup then  $e(G, H) = 0$  if  $G$  has rank 1,  $e(G, H) = 2$  if  $G$  has rank 2 and  $e(G, H) = 1$  if  $G$  has rank greater than or equal to 3.

This invariant generalises Stallings' definition of the number of ends of the group  $G$  since if  $H = \{1\}$  then  $e(G, H) = e(G)$ .

In [4] Stallings showed that the group  $G$  splits over some finite subgroup  $C$  if and only if  $e(G) \geq 2$ . There are several important generalisations of this fact, the most wide ranging being the algebraic torus theorem, established by Dunwoody and Swenson [1]. This states that, under suitable additional hypotheses, if  $G$  contains a polycyclic-by-finite subgroup  $H$  of Hirsch length  $n$  with  $e(G, H) \geq 2$  then either

- (1)  $G$  is virtually polycyclic of Hirsch length  $n + 1$ ,
- (2)  $G$  splits over a virtually polycyclic subgroup of Hirsch length  $n$ , or
- (3)  $G$  is an extension of a virtually polycyclic group of Hirsch length  $n - 1$  by a Fuchsian group.

This theorem generalises the classical torus theorem from low-dimensional topology which asserts that a closed 3-manifold which admits an immersed incompressible torus either admits an embedded incompressible torus or has a Seifert fibration. These topological conclusions imply the algebraic conclusions for the fundamental group of the manifold.

An important ingredient of the proof of the algebraic torus theorem is a special case of the so-called Kropholler conjecture. Its original formulation relies on the following observation of Scott:

A subgroup  $H$  of a finitely generated group  $G$  satisfies  $e(G, H) \geq 2$  if and only if  $G$  admits a subset  $A$  satisfying the following:

- (1)  $A = HA$ ,
- (2)  $A$  is  $H$ -almost invariant, and
- (3)  $A$  is  $H$ -proper, i.e., neither  $A$  nor  $G - A$  is  $H$ -finite.

We will refer to the subset  $A$  as a *proper  $H$ -almost invariant subset*.

In his proof of the algebraic torus theorem for Poincaré duality groups Kropholler observed that, under certain additional hypotheses, if  $G$  admits a proper  $H$ -almost invariant subset  $A$  such that  $A = AH$ , then  $G$  admits a splitting over some subgroup  $C < G$  related to  $H$  (see [2] for an outline of the proof). He conjectured that the additional hypotheses were inessential. Specifically:

**CONJECTURE 53.1** (The Kropholler conjecture). *Let  $G$  be a finitely generated group and  $H < G$ . If  $G$  contains a proper  $H$ -almost invariant subset  $A$  such that  $A = AH$  then  $G$  admits a non-trivial splitting over a subgroup  $C$  which is commensurable with a subgroup of  $H$ .*

The conjecture is known to hold when  $G$  is a Poincaré duality group or when  $G$  is word hyperbolic and  $H$  is a quasi-convex subgroup. In general it is known (for an arbitrary finitely generated group  $G$ ) whenever  $H$  is a subgroup which satisfies the following descending chain condition:

Every descending chain of subgroups  $H = H_0 \geq H_1 \geq H_2 \geq \dots$  such that  $H_{i+1}$  has infinite index in  $H_i$  eventually terminates.

This condition holds for example for the class of finitely generated polycyclic groups, in which class the Hirsch length is the factor controlling the length of such a chain. This is a key ingredient in the proof of the full algebraic torus theorem.

An alternative, more geometric, point of view on the conjecture is provided by the following characterisation:

**THEOREM 53.2.** *Given a finitely generated group  $G$  and a subgroup  $H < G$  the invariant  $e(G, H)$  is greater than or equal to 2 if and only if  $G$  acts with no global fixed point on a  $CAT(0)$  cubical complex with one orbit of hyperplanes, and so that  $H$  is a hyperplane stabiliser.  $H$  admits a right invariant, proper  $H$ -almost invariant subset if and only if the action can be chosen so that  $H$  has a fixed point in the complex.*

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