Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 54 (2008)

Heft: 1-2

Artikel: The uniform Kazhdan property for SLn(Z), n3

Autor: Arzhantseva, Goulnara N.

DOI: https://doi.org/10.5169/seals-109872

L'Enseignement Mathématique (2) 54 (2008), 11–12

3

THE UNIFORM KAZHDAN PROPERTY FOR $\mathrm{SL}_n(\mathbf{Z})$, $n \geq 3$

by Goulnara N. ARZHANTSEVA

Let Γ be a discrete group, and let S be a finite subset of Γ. For a unitary representation π of Γ in a separable Hilbert space $\mathcal{H}$ we define the number

$$K(\pi, \Gamma, S) = \inf_{0 \neq u \in \mathcal{H}} \max_{s \in S} \frac{\|\pi(s)u - u\|}{\|u\|} .$$

Then the *Kazhdan constant* of Γ with respect to S is defined as

$$K(\Gamma, S) = \inf_{\pi} K(\pi, \Gamma, S) ,$$

where the infimum is taken over unitary representations π having no invariant vectors. We also define the *uniform Kazhdan constant* of Γ as

$$K(\Gamma) = \inf_{S} K(\Gamma, S) ,$$

where the infimum is taken over all finite generating sets S of Γ.

A group Γ is said to have *Kazhdan property* (T) (or to be a *Kazhdan group*) if there exists a finite subset S of Γ with $K(\Gamma, S) > 0$. A group Γ is *uniform Kazhdan* if $K(\Gamma) > 0$.

Shortly after its introduction by David Kazhdan in the mid 60's, property (T) was used by Gregory Margulis to give a first explicit construction of infinite families of expander graphs of bounded degree. In particular, a major problem of practical application in the design of efficient communication networks was solved.

A classical example of a Kazhdan group is the group $\mathrm{SL}_n(\mathbf{Z})$ for $n \geq 3$ (for more details and a general context of locally compact groups see the recent book [2]). Surprisingly, the following question is still open.

QUESTION 3.1. *Is the group* $\mathrm{SL}_n(\mathbf{Z})$, *for* $n \geq 3$, *uniform Kazhdan ?*

Infinite finitely generated uniform Kazhdan groups were discovered very recently, see [6] or [1]. However, these groups are neither finitely presented nor residually finite. The latter construction provides an infinite uniform Kazhdan group that weakly (see [4]) contains an infinite family of expanders in its Cayley graph.

An affirmative answer to the above question would give, in particular, the first example of a residually finite (and, in addition, finitely presented) infinite uniform Kazhdan group. It is crucial for applications: infinite families of expanders could be constructed independently of the choice of the group generating set.

A negative answer would be interesting as well. In that case, this classical group would belong to the class of non-uniform Kazhdan groups. First examples of such groups were obtained using Lie groups (see [3]). Then, all word hyperbolic groups were also shown to have zero uniform Kazhdan constant (see [5]).

REFERENCES

[1] ARZHANTSEVA, G. and T. DELZANT. A random group with uniform Kazhdan property. *Preliminary version.*

[2] BEKKA, B., P. DE LA HARPE and A. VALETTE. Kazhdan's Property (T). New Mathematical Monographs *11*. Cambridge University Press, Cambridge, 2008.

[3] GELANDER, T. and A. ZUK. Dependence of Kazhdan constants on generating subsets. *Israel J. Math. 129* (2002), 93–98.

[4] GROMOV, M. Random walk in random groups. *Geom. Funct. Anal. 13* (2003), 73–146.

[5] OSIN, D. Kazhdan constants of hyperbolic groups. *Funktsional. Anal. i Prilozhen. 36* (2002), 46–54; translation in *Funct. Anal. Appl. 36* (2002), 290–297.

[6] OSIN, D. and D. SONKIN. Uniform Kazhdan groups. Preprint arXiv: math.GR/0606012 (2006).

Goulnara N. Arzhantseva

Département de Mathématiques
Université de Genève
2-4, rue du Lièvre
CH-1211 Genève 4
Switzerland
e-mail: goulnara.arjantseva@math.unige.ch